Section 2.3 Objectives

- Use the inequality symbols to compare two numbers.
- Determine if a given value is a solution of an inequality.
- Solve simple inequalities.
- Graph the solutions to inequalities on the number line.
- Write the solution of an inequality using interval notation.
- Solve application word problems by writing algebraic inequalities and solving them.

SECTION 2.3 Inequalities

INTRODUCTION

In the last two sections you learned to solve mathematical statements called ***equations***. The statements were called equations because they contained the equal sign. In this section you will focus on a different kind of mathematical statement called an ***inequality***. An inequality does not contain an equal sign. Instead, it contains a symbol called an inequality symbol. You begin this section by getting familiar with the inequality symbols. After that you will learn to solve and graph inequalities.

INEQUALITY SYMBOLS

An ***inequality symbol*** is used to compare the values of two numbers. For instance, an inequality symbol can indicate if a number is smaller or larger than another number. There are four different inequality symbols. The chart below shows all four inequality symbols and gives the meaning of each one.

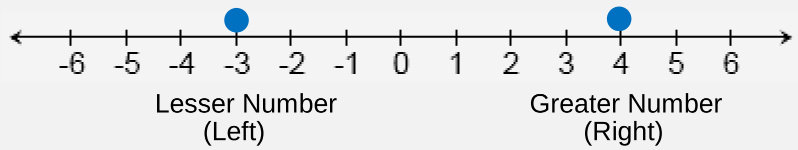
INEQUALITY SYMBOLS			
$<$	less than	$>$	greater than
$\leq$	less than or equal to	$\geq$	greater than or equal to

When an inequality symbol is used in a mathematical statement to compare two numbers, the SMALL (pointed) end of the symbol should point to the smaller number and the LARGE (open) end of the symbol should open to the larger number.

INEQUALITY STATEMENT	
$small < LARGE$	$LARGE > small$
$2 < 6$	$6 > 2$
Read "2 ***is less than*** 6"	Read "6 ***is greater than*** 2"

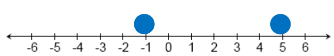
When you compare two numbers, if you have difficulty determining which number is the smaller number and which is the larger, it may help to think of the location of the numbers on a number line. Remember that the numbers increase from left to right on the number line. So the smaller number is to the left, and the larger number is to the right. This method may be especially helpful when you are comparing negative numbers.

USING THE NUMBER LINE TO COMPARE TWO NUMBERS
The <u>smaller/lesser</u> number is always to the <u>left</u>. The <u>larger/greater</u> number is always to the <u>right</u>.



<u>EXAMPLES</u>: Use the < or > symbol between the numbers to make each statement true.

1. $-1 \ \square \ 5$



$-1 < 5$

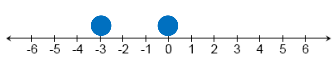
The first number given in the problem is –1.

On the number line, –1 is to the <u>left</u> of 5. This means "–1 is <u>less than</u> 5."

Place the ***less than*** symbol < between the numbers to complete the inequality.

Note: the small (pointed) end of the inequality symbol points to the <u>smaller</u> number –1.

2. $0 \ \square \ -3$



$0 > -3$

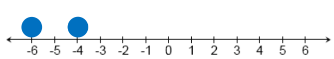
The first number given in the problem is 0.

On the number line, 0 is to the <u>right</u> of –3. This means "0 is <u>greater than</u> –3."

Place the ***greater than*** symbol > between the numbers to complete the inequality.

Note: the large (open) end of the inequality symbol opens to the <u>larger</u> number 0.

3. $-6 \ \square \ -4$



$-6 < -4$

The first number given in the problem is –6.

On the number line, –6 is to the <u>left</u> of –4. This means "–6 is <u>less than</u> –4.

Place the ***less than*** symbol < between the numbers to complete the inequality.

Note: the small (pointed) end of the inequality symbol points to the <u>smaller</u> number –6

4. $(-2)(-7) \square (-8)(3)$

$14 \ \square \ -24$

$14 > -24$

Begin by simplifying each side of the inequality.

On a number line *(not shown here)*, 14 is to the <u>right</u> of –24.
This means "14 is <u>greater than</u> –24.

Place the ***greater than*** symbol > between the numbers to complete the inequality.

Note: the large (open) end of the inequality symbol opens to the <u>larger</u> number 14.

PRACTICE: Use the $<$ or $>$ symbol between the numbers to make each statement true.

1. $5 \;\square\; 4$
2. $-2 \;\square\; 1$
3. $-7 \;\square\; -1$
4. $(-1)(-6) \square (3)(-5)$
5. $7 \;\square\; -3$
6. $-4 \;\square\; 0$
7. $-2 \;\square\; -8$
8. $\frac{3}{2} \;\square\; \frac{1}{2}$

Answers:

1. $5 > 4$
2. $-2 < 1$
3. $-7 < -1$
4. $6 > -15$
5. $7 > -3$
6. $-4 < 0$
7. $-2 > -8$
8. $\frac{3}{2} > \frac{1}{2}$

SOLUTIONS OF INEQUALITIES

We will continue to study inequalities. But now the statements will contain a variable as well. An example is $x > 1$. Like equations, the solution of an inequality is the value of the variable that makes the statement true.

To determine whether a specific value is a solution of an inequality, we use the same method that we used with equations: replace the variable with the given value and determine if the resulting statement is true or false. For instance, if we replace x with 6 in the statement $x > 1$, we get $6 > 1$ which is true. So, we say that 6 is a solution of the inequality.

But with inequalities, more than one value can make the statement true. In fact, an inequality can have an infinite (unlimited) number of solutions. For the inequality $x > 1$, all numbers larger than 1 are solutions. The collection of all solutions of an inequality is called the ***solution set***.

SOLUTION OF AN INEQUALITY
To determine if a given value is a solution of an inequality: 1. Substitute the value in place of each variable in the inequality. 2. Perform the arithmetic on each side of the inequality. • Be sure to follow the proper order of operations (PEMDAS). • Simplify until there is just one number on each side of the inequality. 3. If the resulting statement is true, then the given value is a solution. Otherwise, the given value is not a solution.

EXAMPLES: Determine if the given value is a solution of the inequality.

1. Is 3 a solution of $2x < 10$?

$2x < 10$	Replace x with 3.
$2(\mathbf{3}) < 10$	Simplify the left side of the inequality.
$6 < 10$	This statement says, "6 is less than 10." This is a <u>true</u> statement. So, 3 is a solution.

2. Is -6 a solution of $2x + 5 > x$?

$2x + 5 > x$	Replace each x with –6.
$2(\mathbf{-6}) + 5 > \mathbf{-6}$	Simplify the left side of the inequality.
$-12 + 5 > -6$	
$-7 > -6$	This statement says, " –7 is greater than –6" This is a <u>false</u> statement. So, –6 is <u>not</u> a solution.

3. Is 4 a solution of $x - 10 \geq -5x + 14$?

$x - 10 \geq -5x + 14$	Replace each x with 4.
$\mathbf{4} - 10 \geq -5(\mathbf{4}) + 14$	Simplify each side of the inequality.
$4 + -10 \geq -20 + 14$	
$-6 \geq -6$	This statement says, "–6 is ***greater than <u>or equal to</u>*** –6." This is a <u>true</u> statement. So, 4 is a solution.

PRACTICE: Determine if the given value is a solution of the inequality.

1. Is 4 a solution of $x - 6 > 1$?
2. Is -3 a solution of $2x + 7 < x + 5$?
3. Is -8 a solution of $3x + 24 \leq 0$?
4. Is 6 a solution of $2x - 5 \geq x + 2$?

Answers:

1. No
2. Yes
3. Yes
4. No

Graphs of Inequalities

A couple pages ago, we discussed the solution to the inequality $x > 1$. We verified that 6 is a solution since 6 is greater than 1. We also mentioned that other numbers were solutions as well. For example:

- 4 is a solution since 4 is greater than 1
- 15 is a solution since 15 is greater than 1
- 3.5 is a solution since 3.5 is greater than 1
- $7\frac{1}{2}$ is a solution since $7\frac{1}{2}$ is greater than 1

In fact, there are infinitely many numbers greater than 1. So, we conclude that the ***solution set*** of the inequality $x > 1$ consists of all numbers greater than 1.

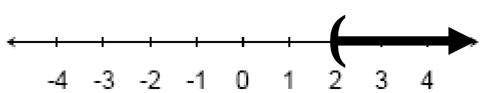
Since we cannot list all the numbers that solve an inequality, a way to represent the numbers visually is to graph the solution set on a number line.

Study the two problems in the boxes below. Compare the two inequalities and their graphs and note the differences between them.

Inequality: $x > 2$

Solution Set: all numbers greater than 2, but not including 2

Graph:



The **parenthesis** on the number line:

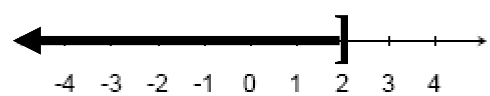
- marks the endpoint at 2
- shows that 2 is not included in the solution

The dark **arrow** is drawn to the right over all the numbers greater than 2.

Inequality: $x \leq 2$

Solution Set: all numbers less than 2, and including 2

Graph:



The **bracket** on the number line:

- marks the endpoint at 2
- shows that 2 is included in the solution

The dark **arrow** is drawn to the left over all the numbers less than 2.

So, to graph an inequality, you will place a mark on the number line at the number given in the problem. The mark represents the endpoint. The mark will either be a parenthesis or a bracket depending on which inequality symbol is in the problem. A parenthesis is used for inequalities containing $<$ or $>$ to indicate that the endpoint is not included in the solution. A bracket is used for inequalities containing $\leq$ or $\geq$ to indicate that the endpoint is included in the solution.

Next, you will draw an arrow starting at the endpoint (parenthesis or bracket) and extending all the way to the left end or right end of the number line. The arrow indicates that the solutions continue endlessly. The direction of the arrow, left or right, depends on which inequality symbol is in the problem. The arrow is drawn to the left for inequalities containing $<$ or $\leq$ to indicate that the solution includes all numbers less than the endpoint. The arrow is drawn to the right for inequalities containing $>$ or $\geq$ to indicate that the solution includes all numbers greater than the endpoint.

The following box summarizes the procedure for graphing inequalities. Remember that the graph depends not only on the number in the problem, but also which inequality symbol is used in the problem. Also, before graphing, it is important for the problem to be set up with the variable to the left of the inequality symbol and the number to the right of the inequality symbol (example: $x > 3$).

GRAPHING INEQUALITIES ON A NUMBER LINE

Make sure the inequality is set up with the variable to the left of the inequality symbol (ex: $x > 3$).

1. ENDPOINT: Place a mark on the number line at the value given in the problem. The inequality symbol determines whether the mark is a parenthesis or bracket.

Parenthesis		Bracket	
$<$ $>$	*means endpoint is NOT included in solution*	$\leq$ $\geq$	*means endpoint is included in solution*

2. ARROW: From the endpoint, draw an arrow. The inequality symbol determines whether the arrow is drawn to the left or right.

Left	Right
$<$	$>$
$\leq$	$\geq$

Helpful Hint: The arrow points in the same direction as the inequality symbol.

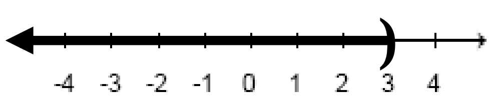
EXAMPLES: Graph each inequality.

1. $x < 3$

Endpoint: The $<$ symbol in the problem means that 3 is not included in the solution. To indicate this on the graph, place a parenthesis on 3.

Arrow: The $<$ symbol in the problem means the solution is all numbers less than 3. To indicate this on the graph, start at 3 and draw an arrow to the left.

Graph:



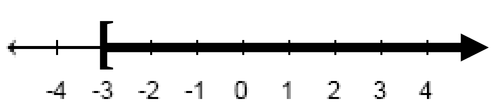
Hint: The arrow points to the left just like the $<$ symbol.

2. $x \geq -3$

Endpoint: The $\geq$ symbol in the problem means that –3 is included in the solution. To indicate this on the graph, place a bracket on –3.

Arrow: The $\geq$ symbol in the problem means the solution is all numbers greater than or equal to –3. To indicate this on the graph, start at –3 and draw an arrow to the right.

Graph:



Hint: The arrow points to the right just like the $\geq$ symbol.

3. $2 \geq x$

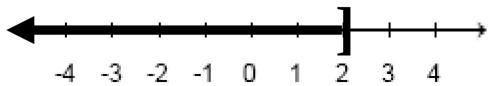
$x \leq 2$

Rewrite: To rewrite the inequality so that the variable is to the left of the inequality symbol:

- Switch what is on the left and right sides of the inequality.
- Switch the direction of the inequality symbol.

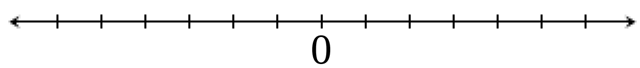
Endpoint: The $\leq$ symbol in the rewritten problem means that 2 is included in the solution. To indicate this on the graph, place a bracket on 2.

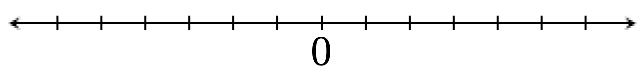
Arrow: The $\leq$ symbol in the rewritten problem means the solution is all numbers less than or equal to 2. To indicate this on the graph, start at 2 and draw an arrow to the left.

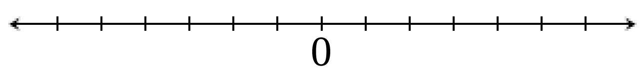
Graph: 

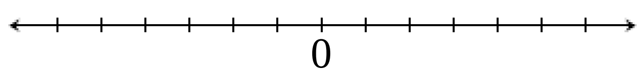
Hint: The arrow points to the left just like the $\leq$ symbol.

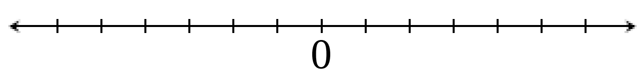
PRACTICE: Graph each inequality.

1. $x > -2$ 

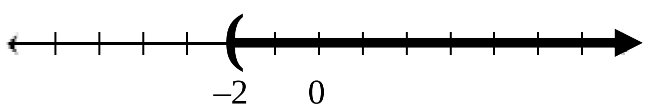
2. $x \leq 4$ 

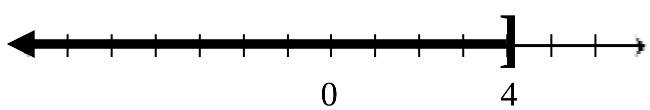
3. $x \geq 1$ 

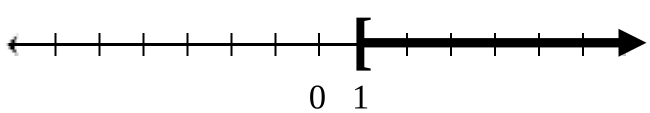
4. $x < -3$ 

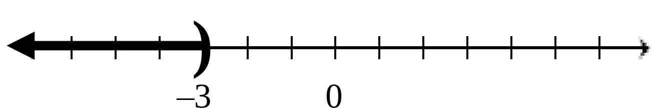
5. $-4 < x$ 

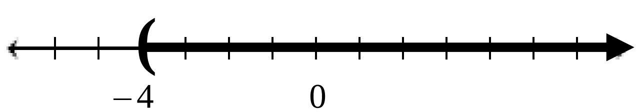
Answers:

1. 

2. 

3. 

4. 

5. 

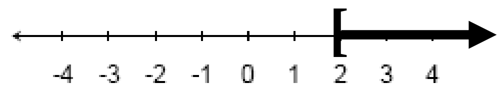
INTERVAL NOTATION

You just learned that the graph of an inequality on a number line is a visual representation of the solution set. The graph shows the interval (part) of the entire number line that contains the solutions to the inequality. The interval shown on the graph can also be described using a special notation called ***interval notation***. Consider the inequality below and the various ways of expressing the solution, including this new interval notation.

Inequality: $x \geq 2$

Solution: All real numbers greater than or equal to 2

Graph:



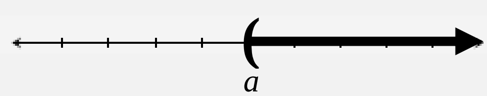
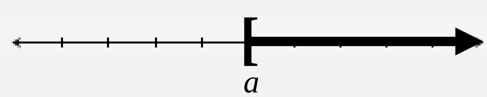
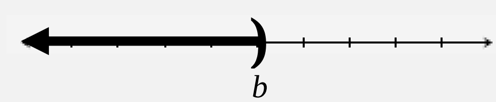
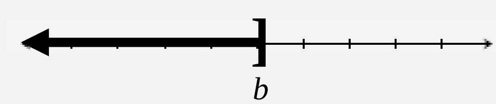
Interval Notation: $[2, \infty)$

Interval notation is a concise way to describe the solution set graphed on the number line.

Interval notation contains two values separated by a comma. The first value is the left endpoint of the graph and gives the smallest value in the solution set. The second value is the right endpoint of the graph and gives the largest value in the solution set. The same endpoint mark (parenthesis or bracket) shown on the graph is also used in interval notation to show whether or not the endpoint is included in the solution set.

Interval notation uses the infinity symbol ∞ if there is no right endpoint on the graph (as in the problem above). It shows that the numbers in the solution set get larger and larger without end. Interval notation uses the negative infinity symbol $-\infty$ if there is no left endpoint on the graph. It shows that the numbers in the solution set get smaller and smaller without end. Interval notation always uses a parenthesis with ∞ and $-\infty$.

The following table shows the four types of inequalities, how they are graphed, and how they are written in interval notation. Notice that the interval notation "matches" the graph.

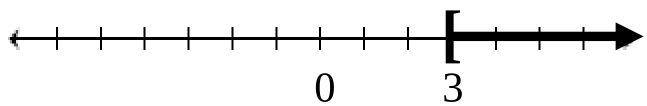
INEQUALITY	GRAPH	INTERVAL NOTATION
$x > a$		(a, ∞)
$x \geq a$		$[a, \infty)$
$x < b$		$(-\infty, b)$
$x \leq b$		$(-\infty, b]$
NOTE: Interval notation always uses a parenthesis with ∞ and $-\infty$.		

EXAMPLES: Graph each inequality and express the solution set in interval notation.

1. $x \geq 3$ Endpoint: The $\geq$ symbol means that 3 is included in the solution.
To indicate this on the graph, place a bracket on 3.

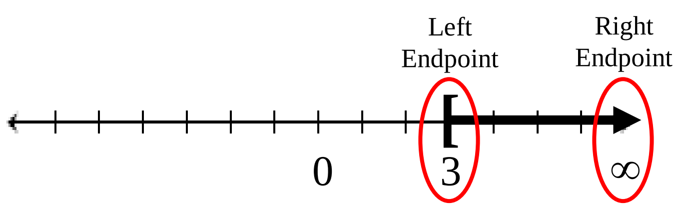
Arrow: The $\geq$ symbol means the solution is all numbers greater than or equal to 3.
To indicate this on the graph, start at 3 and draw an arrow to the right.

Graph:



The smallest value is 3 and the interval extends indefinitely to the right.

Use the graph to write the interval notation:

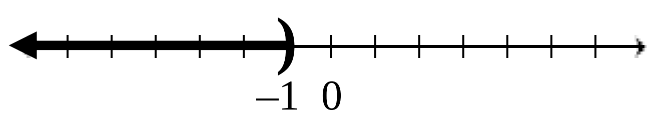


Interval Notation: $[3, \infty)$

2. $x < -1$ Endpoint: The < symbol means that –1 is not included in the solution.
To indicate this on the graph, place a parenthesis on –1.

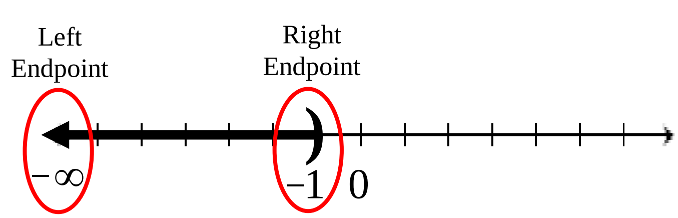
Arrow: The < symbol means the solution is all numbers less than –1.
To indicate this on the graph, start at –1 and draw an arrow to the left.

Graph:



The interval extends indefinitely to the left and the largest value is –1.

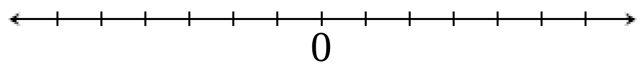
Use the graph to write the interval notation:



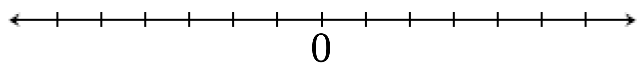
Interval Notation: $(-\infty, -1)$

PRACTICE: Graph each inequality. Write the solution set in interval notation.

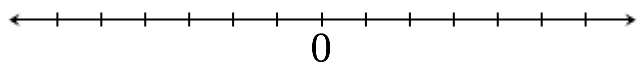
1. $x > 5$



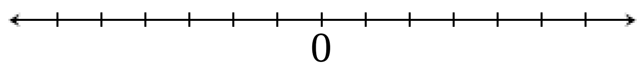
2. $x \leq 1$



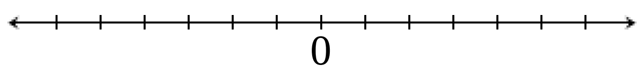
3. $x \geq -4$



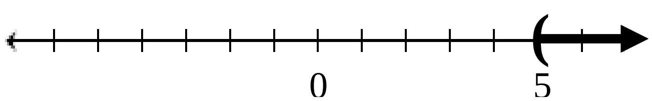
4. $x < -2$



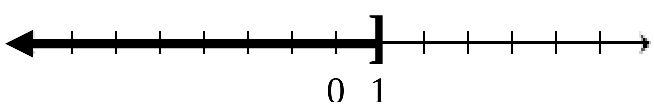
5. $0 < x$ Hint: Begin by rewriting the inequality so that the variable is to the left of the inequality symbol.



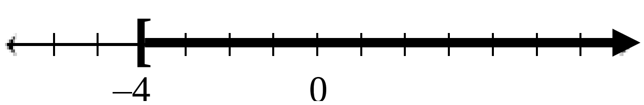
Answers:

1. 

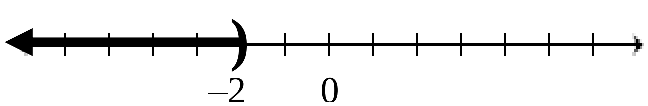
$(5, \infty)$

2. 

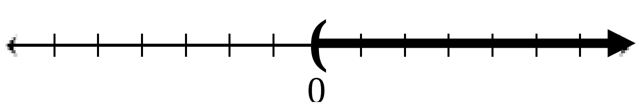
$(-\infty, 1]$

3. 

$[-4, \infty)$

4. 

$(-\infty, -2)$

5. 

$(0, \infty)$

SOLVING INEQUALITIES

All the inequalities you have graphed so far have contained just one variable and just one number. A typical inequality was $x > 4$. But inequalities can be more complex than this. Like equations, they can contain many terms and many operations. An example of an inequality with many terms and operations is $3x - 5 < 7x + 3$.

We will now focus on ***solving inequalities*** such as this. To solve an inequality means to get the variable alone on one side of the inequality symbol. In the case of inequalities, it is recommended that you get the variable alone on the left side of the inequality symbol. This will avoid any confusion when graphing.

To solve an inequality, you will use the same procedures that you used to solve equations with *one important additional algebraic rule*: During the process of solving an inequality, if you multiply or divide both sides of the inequality by a negative number, then you must reverse the direction of the inequality symbol. For instance, the $<$ symbol would change to $>$.

SOLVING AN INEQUALITY
Goal: Get the variable alone on the left side of the inequality symbol. Procedure: Use the same procedures that are used with equations, along with the following additional rule: **Reverse** the direction of the ***inequality symbol*** whenever you **multiply** or **divide** both sides of the inequality by a **negative** number.

EXAMPLES: Solve each inequality.

1. $-9x \geq 108$ — Variable Alone: To get x alone on the left side of the inequality, divide by –9 on both sides.

$$\frac{\cancel{-9}x}{\cancel{-9}} \leq \frac{108}{-9}$$

IMPORTANT: Since we are dividing by a negative number, we must reverse the inequality symbol. We change it from $\geq$ to $\leq$.

$$x \leq -12$$

This is the solution.

2. $4x \cancel{-3} > 9$ — Variable Term Alone: To get $4x$ alone on the left side of the inequality, add 3 to both sides.

$$\underline{\quad \cancel{+3} \quad +3}$$

$$4x > 12$$

Variable Alone: To get x alone on the left side of the inequality, divide by 4 on both sides.

$$\frac{\cancel{4}x}{\cancel{4}} > \frac{12}{4}$$

NOTE: Since we are not dividing by a negative number, the inequality symbol is not reversed.

$$x > 3$$

This is the solution.

3. $-3(2x-3) > -5$ Parentheses: Use the Distributive Property to clear the parentheses.

$$\begin{array}{rcr} -6x \cancel{+9} & > & -5 \\ \cancel{-9} & & -9 \\ \hline \end{array}$$

Variable Term Alone: To get $-6x$ alone on the left side of the inequality, subtract 9 from both sides.

$$-6x > -14$$

Variable Alone: To get x alone on the left side of the inequality, divide by -6 on both sides.

$$\frac{\cancel{-6}x}{\cancel{-6}} < \frac{-14}{-6}$$

IMPORTANT: Since we are dividing by a negative number, we must reverse the inequality symbol. We change it from $>$ to $<$.

$$x < \frac{7}{3}$$

This is the solution.

4. $5x+4 \le \cancel{7x}+14$

$$\begin{array}{rcr} 5x+4 & \le & \cancel{7x}+14 \\ -7x & & \cancel{-7x} \\ \hline \end{array}$$

ONE Variable Term: There are two variable terms: $5x$ and $7x$. With inequalities, we will get the variable term on the left side of the inequality. So to remove $7x$ on the right, subtract $7x$ from both sides. Simplify to get just one variable term on the left side: $-2x$

$$\begin{array}{rcr} -2x+4 & \le & 14 \\ \cancel{-4} & & -4 \\ \hline \end{array}$$

Variable Term Alone: To get $-2x$ alone on the left side, subtract 4 from both sides.

$$-2x \le 10$$

Variable Alone: To get x alone on the left side of the inequality, divide by -2 on both sides.

$$\frac{\cancel{-2}x}{\cancel{-2}} \ge \frac{10}{-2}$$

IMPORTANT: Since we are dividing by a negative number, we must reverse the inequality symbol. We change it from $\le$ to $\ge$.

$$x \ge -5$$

This is the solution.

5. $\frac{x}{-2} > 3$ To eliminate the fraction, multiply both sides of the inequality by **–2**, the LCD.

$$(-2)\left(\frac{x}{-2}\right) < (-2)(3)$$

IMPORTANT: Since we are multiplying by a negative number, we must reverse the inequality symbol. We change it from $>$ to $<$.

$$\left(\frac{\cancel{-2}}{1}\right)\left(\frac{x}{\cancel{-2}}\right) < (-2)(3)$$

On the left side, write **–2** as a fraction, then divide out common factors. Perform the multiplications on each side.

$$x < -6$$

This is the solution.

PRACTICE: Solve each inequality.

1. $-3x \ge -12$
2. $8x+2 \le -46$
3. $2x > 3x-5$
4. $\frac{x}{-9} > 2$
5. $4-2x < 8$
6. $5x+1 \le 2x-8$
7. $4(x-3) < 5x+1$
8. $\frac{2x}{6} \ge 7$

Answers:

1. $x \le 4$
2. $x \le -6$
3. $x < 5$
4. $x < -18$
5. $x > -2$
6. $x \le -3$
7. $x > -13$
8. $x \ge 21$

In this last set of inequality examples, we will combine the major skills that you have learned. We will solve equalities, graph their solution sets, and write the solution sets in interval notation.

EXAMPLES: Solve each inequality, graph the solution set, and write it in interval notation.

1. $-4x+3 \geq 15$

Solution:

$$\begin{array}{rcl} -4x \cancel{+3} & \geq & 15 \\ \underline{\quad \cancel{-3}} & & \underline{-3} \end{array}$$

Variable Term Alone: To get $-4x$ alone on the left side, subtract 3 from both sides.

$$-4x \geq 12$$

Variable Alone: To get x alone on the left side, divide by –4 on both sides.

$$\frac{\cancel{-4}x}{\cancel{-4}} \leq \frac{12}{-4}$$

IMPORTANT: Reverse the inequality symbol since we are dividing by a negative number.

$$x \leq -3$$

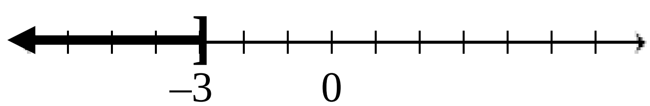
This is the solution as an inequality.

Graph:

The solution set is $x \leq -3$

Endpoint: The $\leq$ symbol means that –3 is included in the solution.
To indicate this on the graph, place a bracket on –3.

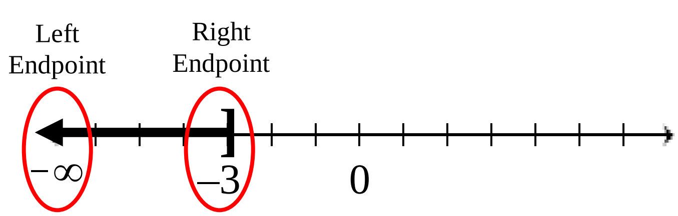
Arrow: The $\leq$ symbol means the solution is all numbers less than or equal to –3.
To indicate this on the graph, start at –3 and draw an arrow to the left.



This is the solution as a graph.

Interval Notation:

Use the graph to write the interval notation:



$(-\infty, -3]$ This is the solution in interval notation.

2. $5-3x\le 2x-15$

Solution:

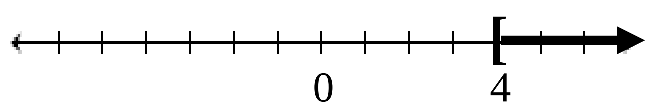
$\begin{array}{r} 5-3x \le \cancel{2x}-15 \\ -2x \;\; \cancel{-2x} \\ \hline \end{array}$	ONE Variable Term: There are two variable terms: $-3x$ and $2x$. With inequalities, we get the variable term on the left side of the inequality. So to remove $2x$ on the right, subtract $2x$ from both sides. Simplify to get just one variable term on the left side: $-5x$
$\begin{array}{r} \cancel{5}-5x\le \quad -15 \\ \cancel{-5} \qquad\quad -5 \\ \hline \end{array}$	Variable Term Alone: To get $-5x$ alone on the left side, subtract 5 from both sides.
$-5x\le -20$	Variable Alone: To get x alone on the left side, divide by -5 on both sides.
$\dfrac{\cancel{-5}x}{\cancel{-5}} \ge \dfrac{-20}{-5}$	IMPORTANT: Reverse the inequality symbol since we are dividing by a negative number.
$x\ge 4$	This is the solution as an inequality.

Graph:

The solution set is $x\ge 4$

Endpoint: The $\ge$ symbol means that 4 is included in the solution.
To indicate this on the graph, place a bracket on 4.

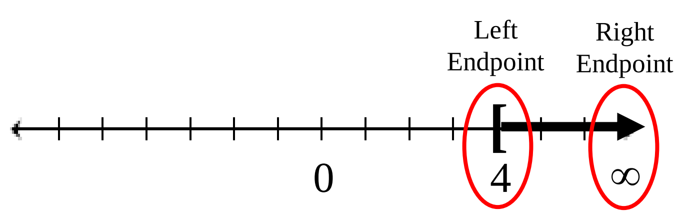
Arrow: The $\ge$ symbol means the solution is all numbers greater than or equal to 4.
To indicate this on the graph, start at 4 and draw an arrow to the right.



This is the solution as a graph.

Interval Notation:

Use the graph to write the interval notation:



$[4,\infty)$ This is the solution in interval notation.

3. $5(x-2)+3<-3(x-1)-2$

Solution:

$5(x-2)+3<-3(x-1)-2$

Parentheses: Use the Distributive Property to clear the parentheses.

$5x\underbrace{-10+3}<\underbrace{-3x+3-2}$

Like Terms: Combine –10 + 3 on the left side and 3 – 2 on the right side.

$$\begin{array}{lcl} 5x \; -7 & < & \cancel{-3x}+1 \\ +3x & & \cancel{+3x} \\ \hline \end{array}$$

ONE Variable Term: There are two variable terms: $5x$ and $-3x$. We will get the variable term on the left side of the inequality. So to remove $-3x$ on the right, add $3x$ to both sides. Simplify to get just one variable term on the left side: $8x$

$$\begin{array}{lcl} 8x \; \cancel{-7} & < & 1 \\ \quad \cancel{+7} & & +7 \\ \hline \end{array}$$

Variable Term Alone: To get $8x$ alone on the left side, add 7 to both sides.

$8x < 8$

Variable Alone: To get x alone on the left side, divide by 8 on both sides.

$$\frac{\cancel{8}x}{\cancel{8}}<\frac{8}{8}$$

NOTE: The inequality symbol is not reversed since we are not dividing by a negative number.

$x<1$

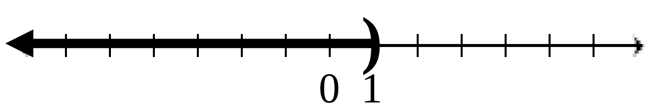
This is the solution as an inequality.

Graph:

The solution set is $x<1$

Endpoint: The $<$ symbol means that 1 is not included in the solution.
To indicate this on the graph, place a parenthesis on 1.

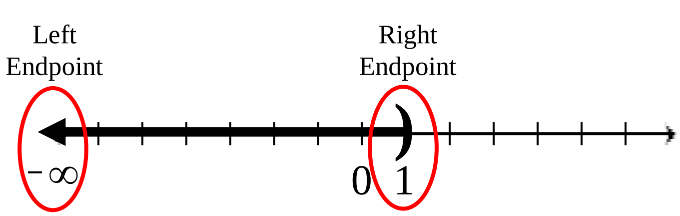
Arrow: The $<$ symbol means the solution is all numbers less than 1.
To indicate this on the graph, start at 1 and draw an arrow to the left.



This is the solution as a graph.

Interval Notation:

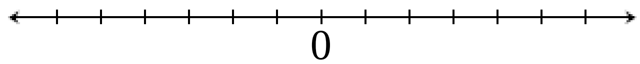
Use the graph to write the interval notation:



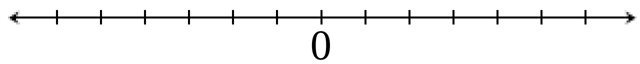
$(-\infty, 1)$ This is the solution in interval notation.

PRACTICE: Solve each inequality, graph the solution set, and write it in interval notation.

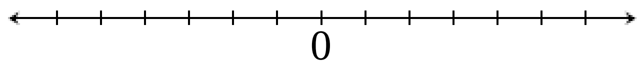
1. $-2x-5<-3$



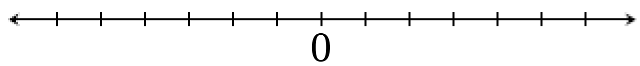
2. $2x+13\geq 7x-12$



3. $4(x+1)<6x+4$



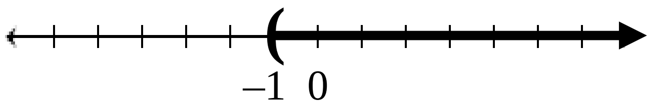
4. $2(3x-4)\geq 2(6-x)+4$



Answers:

1. Solution: $x>-1$

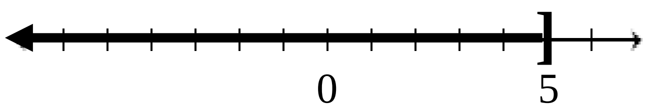
   Graph:



   Interval Notation: $(-1,\infty)$

2. Solution: $x\leq 5$

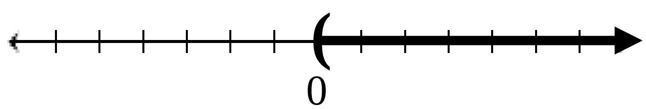
   Graph:



   Interval Notation: $(-\infty,5]$

3. Solution: $x>0$

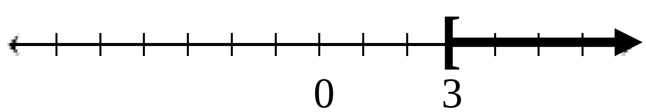
   Graph:



   Interval Notation: $(0,\infty)$

4. Solution: $x\geq 3$

   Graph:



   Interval Notation: $[3,\infty)$

APPLICATION PROBLEMS INVOLVING INEQUALITIES

In previous sections you learned to translate word problems into algebraic ***equations*** and then you solved the equations. You will apply the same concepts now except that you will be writing and solving algebraic ***inequalities*** instead. Make sure you are familiar with the key phrases in the box below so that you can translate them into algebra correctly.

KEY PHRASES			
At least	$\geq$	More than Greater than	$>$
No more than At most	$\leq$	Less than	$<$

EXAMPLES: Write an algebraic inequality for each word problem, then solve the inequality to answer the question.

1. *Six decreased by twice a number is greater than 14. Find the numbers that satisfy this condition.*

Variable: n = the unknown number

Inequality: $\underbrace{\text{6 decreased by}}_{6\,-}\ \underbrace{\text{twice a number}}_{2n}\ \underbrace{\text{is greater than}}_{>}\ \underset{14}{14}$

Solve:

$$\begin{array}{r} 6-2n > 14 \\ \underline{-6 \qquad\ -6} \end{array}$$ To get the variable term alone on the left, subtract 6 from both sides.

$-2n > 8$ To get the variable alone on the left, divide by -2 on both sides.

$\dfrac{\cancel{-2}n}{\cancel{-2}} < \dfrac{8}{-2}$ The inequality symbol is reversed since we are dividing by a negative number.

$n < -4$

Solution: $n < -4$ The solution set consists of all numbers less than -4.

2. *The product of 3 and a number is no more than the sum of 9 and four times the number. Find the numbers that satisfy this condition.*

Variable: n = the unknown number

Inequality: $\underbrace{\text{The product of 3 and a number}}_{3n}\ \underbrace{\text{is no more than}}_{\le}\ \underbrace{\text{the sum of 9 and}}_{9+}\ \underbrace{\text{four times the number}}_{4n}$

Solve:

$$3n \le 9 + 4n$$

$$\begin{array}{r} 3n \le 9 + 4n \\ -4n \quad\ \ -4n \\ \hline \end{array}$$ To remove the variable term $4n$ on the right side of the inequality, subtract $4n$ from both sides.

$$-1n \le 9$$ Simplify to get just one variable term on the left side: $-1n$

$$\frac{\cancel{-1}n}{\cancel{-1}} \ge \frac{9}{-1}$$ To get the variable alone on the left, divide by –1 on both sides.

The inequality symbol is reversed since we are dividing by a negative number.

$$n \ge -9$$

Solution: $n \ge -9$ The solution set consists of all numbers greater than or equal to –9.

3. *A reception hall does not charge a rental fee if at least $3500 is spent on food. For their wedding reception at the hall, a couple plans to serve a dinner that costs $28 per person. How many people must attend the reception for the couple to avoid paying the rental fee?*

Variable: p = the number of people that must attend

Inequality: $\underbrace{\text{cost per person}}_{28} \times \underbrace{\text{\# of people}}_{p}\ \underbrace{\text{is at least}}_{\ge}\ 3500$

Solve:

$$28p \ge 3500$$ To get the variable alone on the left, divide by 28 on both sides.

$$\frac{\cancel{28}p}{\cancel{28}} \ge \frac{3500}{28}$$ The inequality symbol is not reversed since we are not dividing by a negative number.

$$p \ge 125$$

Solution: $p \ge 125$ At least 125 people must attend the reception to avoid the rental fee.

4. *A delivery man has to deliver a truckload of boxes to the top floor of a business. Each box weighs 80 pounds, and the dolly the man will use to transport the boxes weighs 20 pounds. The delivery man himself weighs 180 pounds. If the elevator in the building can hold at most 1500 pounds, how many boxes can be delivered at a time?*

Variable: b = the number of boxes that can be delivered at one time

Inequality:

$$\underbrace{\text{Weight of Man}} + \underbrace{\text{Weight of Dolly}} + \underbrace{\text{Weight of Boxes}}_{\text{(weight of 1 box) (\# of boxes)}} \ \underbrace{\text{is at most}} \ 1500$$

$$180 + 20 + (80)(b) \leq 1500$$

Solve:

$180+20+80b \leq 1500$ Combine the like terms on the left side of the inequality.

$$\begin{array}{r} 200 \ + \ 80b \leq 1500 \\ -200 \qquad\quad -200 \\ \hline \end{array}$$

To get the variable term $80b$ alone on the left side, subtract 200 from both sides.

$80b \leq 1300$ To get b alone on the left side, divide by 80 on both sides.

$\dfrac{80b}{80} \leq \dfrac{1300}{80}$ We are not dividing by a negative number, so we do not reverse the inequality symbol.

$b \leq 16.25$ The answer is between 16 and 17 boxes, but the delivery man cannot deliver part of a box. If he delivers 17 boxes at a time, it will exceed the weight limit. So he must deliver 16 boxes at a time.

Solution: 16 boxes At most, **16 boxes** can be delivered at one time using the elevator.

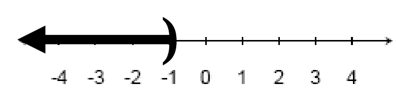
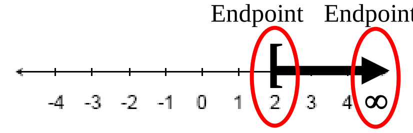
PRACTICE: Write an algebraic inequality for each problem, then solve it to answer the question.

1. The sum of a number and 17 is less than 39. Find the numbers that satisfy this condition.
2. The difference of 8 and a number is at least 24. Find the numbers that satisfy this condition.
3. Three times a number increased by 4 is no more than 16. Find the numbers that satisfy this condition.
4. Six more than twice a number is less than the product of 4 and the number. Find the numbers that satisfy this condition.
5. A number decreased by 7 is greater than the difference of eight times the number and 21. Find the numbers that satisfy this condition.
6. A charity will raffle off a new car donated by a car dealer. The raffle tickets will be sold for $50 each. How many tickets must be sold to raise at least $20,000?
7. The school is having a performance in the auditorium. There are 32 students involved in the performance. Each of these students is allowed to invite an equal number of guests to attend the performance. How many guests can each student invite if the auditorium can hold no more than 240 people, including those in the performance?
8. In four weeks, Tina needs $760 to pay for her tuition. She currently has $600. How much must Tina save from each of her next four paychecks to have at least $760 for her tuition?

Answers:

1. $n+17<39$; $n<22$
2. $8-n \geq 24$; $n \leq -16$
3. $3n+4 \leq 16$; $n \leq 4$
4. $2n+6<4n$; $n>3$
5. $n-7>8n-21$; $n<2$
6. $50r \geq 20{,}000$; At least 400 tickets
7. $32+32f \leq 240$; No more than 6 guests each
8. $600+4p \geq 760$; At least $40

SECTION 2.3 SUMMARY
Inequalities

INEQUALITY SYMBOLS	Used to compare the values of two numbers. $<$ less than $\leq$ less than or equal to $>$ greater than $\geq$ greater than or equal to	*Example*: Use $<$ or $>$ between the numbers to make a true statement. $2 \;\square\; -6$ On a number line, 2 is to the right of –6. So, 2 is greater than –6. Place the $>$ symbol between the numbers. $2 > -6$ Note: The large (open) end of the inequality symbol opens to the larger number 2.
SOLUTION OF AN INEQUALITY	A number that can replace the variable to make the inequality true. 1. Replace each variable with the given value. 2. Simplify each side. 3. If the resulting inequality is true, the value is a solution.	*Example*: Is 2 a solution of $5x > x+6$? $5(2) > 2+6$ $10 > 8$ Yes, 2 is a solution.
GRAPH OF AN INEQUALITY	Shows all the solutions of an inequality on a number line. 1. Make sure the inequality has the variable on the left side. 2. On the graph, put a parenthesis or bracket at the endpoint. **Parenthesis:** $<$ and $>$ — Endpoint is not included in solution **Bracket:** $\leq$ and $\geq$ — Endpoint is included in solution 3. Draw an arrow from the endpoint to the left or right. **Left:** $<$ and $\leq$ **Right:** $>$ and $\geq$	*Example*: Graph $x < -1$ 
INTERVAL NOTATION	Expresses the solution set of an inequality using the endpoints of the interval. 1. List the left endpoint first and the right endpoint second. 2. Use the same symbol (parenthesis or bracket) with the endpoint that is shown with it on the graph. • If there is no right endpoint, use ∞ with a parenthesis. • If there is no left endpoint, use $-\infty$ with a parenthesis.	*Example*: Graph $x \geq 2$ and write the solution in interval notation. Left Endpoint Right Endpoint  Interval Notation: $[2, \infty)$ *"Matches" the graph.*
SOLVING AN INEQUALITY	Determine the values of the variable that make the inequality true. GOAL: Get the variable alone on the **left** side of the inequality. HOW: Use the same methods that were used to solve equations. **Important Additional Rule**: If both sides of the inequality are multiplied or divided by a negative number, the direction of the inequality symbol must be reversed.	*Example*: Solve $-4x + 3 \leq 15$ $ -3 \quad -3$ $-4x \leq 12$ $\frac{\cancel{-4}x}{\cancel{-4}} \geq \frac{12}{-4}$ $x \geq -3$
APPLICATION PROBLEMS	Use a variable for the unknown and use key words to identify the inequality symbol.	$\geq$ At least $\leq$ At most, No more than

SECTION 2.3 EXERCISES

Inequalities

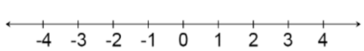
Use the $<$ or $>$ symbol between the numbers to make each statement true.

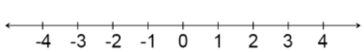
1. $-7 \ \square \ 3$
2. $0 \ \square \ -2$
3. $-8 \ \square \ -9$
4. $(-4)(5) \ \square \ (-1)(-6)$

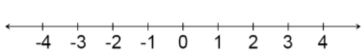
Determine if the given value is a solution of the inequality.

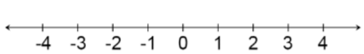
5. Is 1 a solution of $-7x+19<-16$?
6. Is -3 a solution of $9x-2>10x$?
7. Is 5 a solution of $21-x>6+2x$?
8. Is -8 a solution of $-6x+7\geq 15+2x$?
9. Is -4 a solution of $4(x-1)\leq 7x+8$?
10. Is 2 a solution of $9x+7<2(4x-1)$?

Graph each inequality. Write the solution set in interval notation.

11. $x<3$ 

12. $x\geq -2$ 

13. $x>4$ 

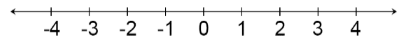
14. $x\leq -1$ 

Solve each inequality.

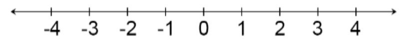
15. $7x\leq -91$
16. $-50x>350$
17. $\frac{x}{4}<-5$
18. $4x+17\geq 37$
19. $6x>8x-20$
20. $5x\geq 6x+7$
21. $2x-1-7x>29$
22. $15x-4\leq -30+11x+14$
23. $2x-54<-x+21$
24. $12-2x\geq 8x+36$
25. $5x\leq 10(3x+5)$
26. $7(x-4)>-7$
27. $7(x+8)\geq 5x+6$
28. $-4(x+12)<2x-6$
29. $4(x-7)\leq 2(x+9)$
30. $3x-4x+5>2(x-2)$
31. $9-(x-8)<x+29$
32. $3+2(x+1)\geq 6+3x$

Solve each inequality, graph the solution set, and write it in interval notation.

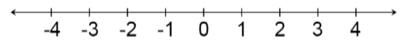
33. $-5x+9\geq 24$



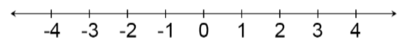
34. $-5x+3<3x+19$



35. $5(12-3x)>15x+60$



36. $2(x+3)+3x\geq 3x+4$



Write an algebraic inequality for each problem, then solve the inequality to answer the question.

37. A number decreased by 5 is less than 33. Find the numbers that satisfy this condition.

38. The sum of a number and 13 is at least 27. Find the numbers that satisfy this condition.

39. Twice a number is more than the sum of that number and 14. Find the numbers that satisfy this condition.

40. The product of 2 and a number is no more than 3 times the number plus 8. Find the numbers that satisfy this condition.

41. Three times a number minus 18 is at least 5 times the number plus 22. Find the numbers that satisfy this condition.

42. The product of 4 and a number is greater than the difference of the number and 21. Find the numbers that satisfy this condition.

43. The boy scouts hope to make at least \$600 on their annual mulch sale. If they make \$3.75 on each bag of mulch that is sold, how many bags must they sell to reach their goal?

44. Sam's doctor recommends that he limit his fat intake to at most 60 grams per day. For breakfast, Sam had 8 grams of fat, and for lunch he had 23 grams of fat. How many grams of fat can Sam have the rest of the day?

45. Mya's bank requires that she maintain a balance of at least \$1500 for free checking. Mya's current balance is \$1670, but she needs to write checks for \$425 and \$173. How much money will Mya have to deposit before writing the checks to maintain the required balance?

46. Darren is selling magazines and will get a prize if he sells more than 150. He has already sold 65 and only has 4 weeks left to sell the magazines. How many must he sell per week to get the prize?

47. To rent a moving truck, a company charges a \$70 fee plus \$15 per day. For how many days can the truck be rented to keep the total cost at no more than \$150?

48. An Olympic speed skater scored times of 6.95 minutes and 7.08 minutes on his first two trials. What time will he need on his third trial so that his average time is less than 7.0 minutes?

Answers to Section 2.3 Exercises

1. $-7 < 3$

2. $0 > -2$

3. $-8 > -9$

4. $(-4)(5) < (-1)(-6)$

5. No

6. Yes

7. No

8. Yes

9. Yes

10. No

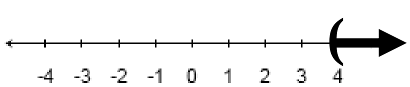
11.

$(-\infty, 3)$

12.

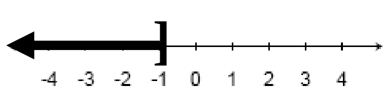
$[-2, \infty)$

13.



$(4, \infty)$

14.



$(-\infty, -1]$

15. $x \le -13$

16. $x < -7$

17. $x < -20$

18. $x \ge 5$

19. $x < 10$

20. $x \le -7$

21. $x < -6$

22. $x \le -3$

23. $x < 25$

24. $x \le -\frac{12}{5}$ OR $x \le -2.4$

25. $x \ge -2$

26. $x > 3$

27. $x \ge -25$

28. $x > -7$

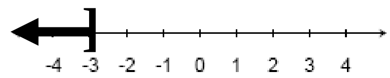
29. $x \le 23$

30. $x < 3$

31. $x > -6$

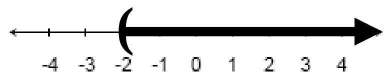
32. $x \le -1$

33. $x \le -3$



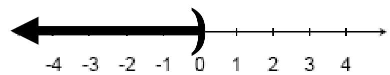
$(-\infty, -3]$

34. $x > -2$



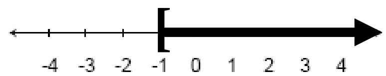
$(-2, \infty)$

35. $x < 0$



$(-\infty, 0)$

36. $x \ge -1$



$[-1, \infty)$

37. Inequality: $n - 5 < 33$ Solution: $n < 38$

38. Inequality: $n + 13 \ge 27$ Solution: $n \ge 14$

39. Inequality: $2n > n + 14$ Solution: $n > 14$

40. Inequality: $2n \le 3n + 8$ Solution: $n \ge -8$

41. Inequality: $3n - 18 \ge 5n + 22$ Solution: $n \le -20$

42. Inequality: $4n > n - 21$ Solution: $n > -7$

43. Inequality: $3.75b \ge 600$ Solution: The boy scouts must sell at least 160 bags of mulch.

44. Inequality: $8 + 23 + r \le 60$ Solution: Sam can have at most 29 grams of fat.

45. Inequality: $1670 - 425 - 173 + d \ge 1500$ Solution: Mya should deposit at least $428.

46. Inequality: $65 + 4m > 150$ Solution: Darren must sell more than 21 magazines.

47. Inequality: $70 + 15d \le 150$ Solution: The truck can be rented for no more than 5 days.

48. Inequality: $\dfrac{6.95 + 7.08 + x}{3} < 7.0$ Solution: The time must be less than 6.97 minutes.

Mixed Review

Sections 1.1 – 2.3

1. Evaluate $\sqrt{100}-|-50|\div(-5)^2\times(-4-3)$.
2. Evaluate $\frac{2}{3}x-\frac{4}{7}y+z$ if $x=6$, $y=\frac{21}{2}$, and $z=-5$.
3. Simplify $-4+6x-5y-9x+10-2y$.
4. Simplify $-\frac{3}{8}x+\frac{3}{5}+\frac{1}{2}x-\frac{1}{4}$.
5. Simplify $-8\left(\frac{5}{4}x-2\right)$.
6. Is $x=4$ a solution of $16-5x=2(1-x)$?
7. Solve $6(2x+3)-10x=-1+5$.
8. Solve $\frac{2}{3}x-8=\frac{5}{6}+\frac{1}{2}x$.
9. Translate the word problem into an algebraic equation. Then solve the equation.

   Six more than the product of -7 and a number is -8. Determine the number.
10. Write an algebraic equation for the word problem. Then solve the equation to answer the question.

    You are buying a used car from your aunt for $6000. You have $1800 to give her now, and she agrees to let you pay the rest in monthly payments. If you pay your aunt $150 per month, how many months will it take to pay back the full amount?

Answers to Mixed Review

1. 24
2. -7
3. $-3x-7y+6$
4. $\frac{1}{8}x+\frac{7}{20}$
5. $-10x+16$
6. No, $-4\neq-6$.
7. $x=-7$
8. $x=53$
9. $n=2$
10. 28 months