Section 4.6: Solving Equations by Factoring

Objective: Solve quadratic equations by factoring and using the zero product rule.

When solving linear equations such as $2x - 5 = 21$; we can solve for the variable directly by adding 5 and dividing by 2, on both sides, to get 13. However, when we have x^2 (or a higher power of x), we cannot just isolate the variable, as we did with the linear equations. One method that we can use to solve for the variable is known as the zero product rule.

Zero Product Rule: If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

The zero product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero. We can use this to help us solve factored polynomials as in the following example.

Example 1. Solve the equation.

$$(2x - 3)(5x + 1) = 0 \quad \text{One factor must be zero}$$

$$2x - 3 = 0 \quad \text{or} \quad 5x + 1 = 0 \quad \text{Set each factor equal to zero}$$

$$\frac{+3 \quad +3}{} \qquad \frac{-1 \quad -1}{} \quad \text{Solve each equation}$$

$$\frac{2x}{2} = \frac{3}{2} \quad \text{or} \quad \frac{5x}{5} = \frac{-1}{5}$$

$$x = \frac{3}{2} \quad \text{or} \quad \frac{-1}{5} \quad \text{Our Solution}$$

For the zero product rule to work, we must have factors to set equal to zero. This means if the problem is not already factored, we will need to factor it first, if at all possible.

Example 2. Solve the equation by factoring.

$$x^2 - 7x + 12 = 0 \quad \text{Multiply to 12; sum to } -7$$

$$(x - 3)(x - 4) = 0 \quad \text{Numbers are } -3 \text{ and } -4.$$

$$x - 3 = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{Since one factor must be zero, set each factor equal to zero}$$

$$\frac{+3 \quad +3}{} \qquad \frac{+4 \quad +4}{} \quad \text{Solve each equation}$$

$$x = 3 \quad \text{or} \quad x = 4 \quad \text{Our Solution}$$

Another important part of the zero product rule is that before we factor, one side of the equation must be zero. If one side of the equation is not zero, we must move terms around so that one side of the equation is zero. Generally, we like the x^2 term to be positive.

Example 3. Solve the equation by factoring.

$$x^2 = 8x - 15$$ Set equal to zero by moving terms to the left
$$\underline{-8x + 15 \quad -8x + 15}$$
$$x^2 - 8x + 15 = 0$$ Factor; multiply to 15; sum to -8
$$(x-5)(x-3) = 0$$ Numbers are -5 and -3
$$x - 5 = 0 \quad \text{or} \quad x - 3 = 0$$ Set each factor equal to zero
$$\underline{+5 \quad +5} \qquad \underline{+3 \quad +3}$$ Solve each equation
$$x = 5 \quad \text{or} \quad x = 3$$ Our Solutions

Example 4. Solve the equation by factoring.

$$(x-7)(x+3) = -9$$ Not equal to zero; multiply first; use FOIL
$$x^2 - 7x + 3x - 21 = -9$$ Combine like terms
$$x^2 - 4x - 21 = -9$$ Move -9 to other side so equation equals zero
$$\underline{+9 \quad +9}$$
$$x^2 - 4x - 12 = 0$$ Factor; multiply to -12; sum to -4
$$(x-6)(x+2) = 0$$ Numbers are -6 and $+2$
$$x - 6 = 0 \quad \text{or} \quad x + 2 = 0$$ Set each factor equal to zero
$$\underline{+6 \quad +6} \qquad \underline{-2 \quad -2}$$ Solve each equation
$$x = 6 \quad \text{or} \quad x = -2$$ Our Solution

Example 5. Solve the equation by factoring.

$$3x^2 + 4x - 5 = 7x^2 + 4x - 14$$ Set equal to zero by moving terms to the right side of the
$$\underline{-3x^2 - 4x + 5 \quad -3x^2 - 4x + 5}$$ equal sign
$$0 = 4x^2 - 9$$ Factor using the difference of two squares
$$0 = (2x+3)(2x-3)$$ One factor must be zero
$$2x + 3 = 0 \quad \text{or} \quad 2x - 3 = 0$$ Set each factor equal to zero
$$\underline{-3 \quad -3} \qquad \underline{+3 \quad +3}$$
$$\dfrac{2x}{2} = \dfrac{-3}{2} \quad \text{or} \quad \dfrac{2x}{2} = \dfrac{3}{2}$$ Solve each equation

$$x = -\frac{3}{2} \quad \text{or} \quad \frac{3}{2} \qquad \text{Our Solutions}$$

Most problems with x^2 will have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 6. Solve the equation by factoring.

$$
\begin{array}{ll}
4x^2 = 12x - 9 & \text{Set equal to zero by moving terms to the left} \\
\underline{-12x+9 \quad -12x+9} & \\
4x^2 - 12x + 9 = 0 & \text{Factor; } 4x^2 = (2x)^2; 12x = 2(2x)(3); 9 = (3)^2 \\
(2x-3)^2 = 0 & \text{Perfect square trinomial; use square roots from first and last terms and sign from the middle} \\
2x - 3 = 0 & \text{Set this factor equal to zero} \\
\underline{+3 \quad +3} & \text{Solve the equation} \\
\dfrac{2x}{2} = \dfrac{3}{2} & \\
x = \dfrac{3}{2} & \text{Our Solution}
\end{array}
$$

As always, it will be important to factor out the GCF first, if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This may give us more than just two solutions. The next few examples illustrate this.

Example 7. Solve the equation by factoring.

$$
\begin{array}{ll}
4x^2 = 8x & \text{Set equal to zero by moving the } 8x \text{ to the left side of the} \\
\underline{-8x \quad -8x} & \text{equal sign} \\
4x^2 - 8x = 0 & \\
4x(x-2) = 0 & \text{One factor must be zero}
\end{array}
$$

$$
\begin{array}{ccc}
\dfrac{4x}{4} = \dfrac{0}{4} & \text{or} & x - 2 = 0 \qquad \text{Set each factor equal to zero} \\
& & \underline{+2 \quad +2} \qquad \text{Solve each equation} \\
x = 0 & \text{or} & 2 \qquad \text{Our Solution}
\end{array}
$$

Example 8. Solve the equation by factoring.

$$
\begin{array}{ll}
2x^3 - 14x^2 + 24x = 0 & \text{Factor out the GCF of } 2x \\
2x(x^2 - 7x + 12) = 0 & \text{Multiply to } 12 \text{; sum to } -7 \\
2x(x-3)(x+4) = 0 & \text{Numbers are } -3 \text{ and } -4
\end{array}
$$

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$$\begin{array}{ccc} 2x=0 & x-3=0 & x-4=0 \\ \dfrac{2x}{2} \dfrac{0}{2} \quad \text{or} & \underline{+3 \ +3} \quad \text{or} & \underline{+4 \ +4} \end{array}$$
Set each factor equal to zero
Solve each equation

$$x=0 \quad \text{or} \quad 3 \quad \text{or} \quad 4 \qquad \text{Our Solutions}$$

Example 9. Solve the equation by factoring.

$$3x^2 + 3x - 60 = 0 \qquad \text{Factor out the GCF of 3}$$
$$3(x^2 + x - 20) = 0 \qquad \text{Multiply to } -20; \text{ sum to 1}$$
$$3(x+5)(x-4) = 0 \qquad \text{Numbers are 5 and } -4$$
$$3(x+5)\,(x-4) \qquad \text{One factor must be zero}$$
$$3 = 0 \quad \text{or} \quad x+5=0 \quad \text{or} \quad x-4=0 \qquad \text{Set each factor equal to zero}$$
$$3 \neq 0 \qquad \underline{-5 \ -5} \qquad \underline{+4 \ +4} \qquad \text{Solve each equation}$$
$$x = -5 \quad \text{or} \quad x = 4$$
$$x = -5 \quad \text{or} \quad x = 4 \qquad \text{Our Solutions}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we get a false statement. No solution comes from this factor. Often a student will skip setting the GCF factor equal to zero if there are no variables in the GCF, which is acceptable.

Just as not all polynomials can be factored, not all equations can be solved by factoring. If an equation does not factor we will have to solve it using another method. These other methods are saved for another course.

World View Note: While factoring works great to solve problems with x^2, Tartaglia, in 16th century Italy, developed a method to solve problems with x^3. He kept his method a secret until another mathematician, Cardano, talked him out of his secret and published the results. To this day the formula is known as Cardano's Formula.

A question often asked is if it is possible to get rid of the square on the variable by taking the square root of both sides. While it is possible, there are a few properties of square roots that will not be covered in this course, and thus it is common to break a rule of roots that we are not aware of at this point. The short reason we want to avoid this for now is because taking a square root will only allow us to consider the principal or positive square root. For now, **never** take the square root of both sides!

4.6 Practice

Solve each equation by factoring.

1) $(k-7)(k+2)=0$

2) $(a+4)(a-3)=0$

3) $(x-1)(x+4)=0$

4) $0=(2x+5)(x-7)$

5) $6x^2-150=0$

6) $p^2+4p-32=0$

7) $2n^2+10n-28=0$

8) $m^2-m-30=0$

9) $x^2-4x-8=-8$

10) $v^2-8v-3=-3$

11) $x^2-5x-1=-5$

12) $a^2-6a+6=-2$

13) $7r^2+84=-49r$

14) $7m^2-224=28m$

15) $x^2-6x=16$

16) $7n^2-28n=0$

17) $3v^2=5v$

18) $2b^2=-3b$

19) $9x^2=30x-25$

20) $3n^2+39n=-36$

21) $4k^2+18k-23=6k-7$

22) $a^2+7a-9=-3+6a$

23) $9x^2-46+7x=7x+8x^2+3$

24) $x^2+10x+30=6$

25) $2m^2+19m+40=-5m$

26) $5n^2+45n+38=-2$

27) $5x(3x-6)-5x^2=x^2+6x+45$

28) $8x^2+11x-48=3x$

29) $41p^2+183p-196=183p+5p^2$

30) $121w^2+8w-7=8w-6$

4.6 Answers

1) 7, −2
2) −4, 3
3) 1, −4
4) $-\frac{5}{2}$, 7
5) −5, 5
6) 4, −8
7) 2, −7
8) −5, 6
9) 0, 4
10) 0.8
11) 1, 4
12) 2, 4
13) −4, −3
14) −4, 8
15) −2, 8
16) 0, 4
17) 0, $\frac{5}{3}$
18) $-\frac{3}{2}$, 0
19) $\frac{5}{3}$
20) −12, −1
21) −4, 1
22) 2, −3
23) −7, 7
24) −4, −6
25) −10, −2
26) −8, −1
27) −1, 5
28) −3, 2
29) $-\frac{7}{3}$, $\frac{7}{3}$
30) $-\frac{1}{11}$, $\frac{1}{11}$