# Section 3.1: Square Roots

## Objective: Simplify expressions with square roots.

To reverse the process of squaring a number, we find the square root of a number. In other words, a square root "un-squares" a number.

**PRINCIPAL SQUARE ROOT**

If $a$ is a n*onnegative* real number, then the **principal square root** of $a$ is the *nonnegative* number $b$ such that $b^2 = a$.
We write $b = \sqrt{a}$.

The symbol $\sqrt{\ }$ is called the *radical sign* and the number $a$, under the radical sign, is called the *radicand*. An expression containing a radical sign is called a *radical expression*. Square roots are the most common type of radical expressions used.

The following example shows several square roots:

**Example 1.** Evaluate.

| | |
|---|---|
| $\sqrt{1} = 1$ because $1^2 = 1$ | $\sqrt{121} = 11$ because $11^2 = 121$ |
| $\sqrt{4} = 2$ because $2^2 = 4$ | $\sqrt{625} = 25$ because $25^2 = 625$ |
| $\sqrt{9} = 3$ because $3^2 = 9$ | $\sqrt{0} = 0$ because $0^2 = 0$ |
| $\sqrt{16} = 4$ because $4^2 = 16$ | $\sqrt{81} = 9$ because $9^2 = 81$ |
| $\sqrt{25} = 5$ because $5^2 = 25$ | $\sqrt{-81}$ is not a real number |

Notice that $\sqrt{-81}$ is not a real number because there is no real number whose square is $-81$. If we square a positive number or a negative number, the result will always be positive. Thus, we can only take square roots of nonnegative numbers. In another section, we will define a method we can use to work with and evaluate square roots of negative numbers, but for now we will state they are not real numbers.

We call numbers like 1, 4, 9, 16, 25, 81, 121, and 625 *perfect squares* because they are squares of integers. Not all numbers are perfect squares. For example, 8 is not a perfect square because 8 is not the square of an integer. Using a calculator, $\sqrt{8}$ is approximately equal to 2.828427125... and that number is still a rounded approximation of the square root.

## SIMPLIFYING SQUARE ROOTS

Instead of using decimal approximations, we will usually express roots in *simplest* radical form. Advantages of simplest radical form are that it is an exact answer (not an approximation) and that calculations and algebraic manipulations can be done more easily.

To express roots in simplest radical form, we will use the following property:

**PRODUCT RULE OF SQUARE ROOTS**

For any *nonnegative* real numbers $a$ and $b$,

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$$

When simplifying a square root expression, we will first find the largest perfect square factor of the radicand. Then, we will write the radicand as the product of two factors, apply the product rule, and evaluate the square root of the perfect square factor.

**Example 2.** Simplify.

| | |
|---|---|
| $\sqrt{75}$ | $75$ is divisible by the perfect square $25$; Split radicand into factors |
| $=\sqrt{25\cdot 3}$ | Apply product rule |
| $=\sqrt{25}\cdot\sqrt{3}$ | Take the square root of $25$ |
| $=5\sqrt{3}$ | Our Answer |

If there is a coefficient in front of the radical to begin with, the problem becomes a big multiplication problem.

**Example 3.** Simplify.

| | |
|---|---|
| $5\sqrt{63}$ | $63$ is divisible by the perfect square $9$; Split radicand into factors |
| $=5\sqrt{9\cdot 7}$ | Apply product rule |
| $=5\sqrt{9}\cdot\sqrt{7}$ | Take the square root of $9$ |
| $=5\cdot 3\sqrt{7}$ | Multiply coefficients |
| $=15\sqrt{7}$ | Our Answer |

As we simplify radicals using this method, it is important to be sure our final answer can be simplified no more.

**Example 4.** Simplify.

| | |
|---|---|
| $\sqrt{72}$ | $72$ is divisible by the perfect square $9$; Split radicand into factors |
| $=\sqrt{9\cdot 8}$ | Apply product rule |
| $=\sqrt{9}\cdot\sqrt{8}$ | Take the square root of $9$ |
| $=3\sqrt{8}$ | But $8$ is also divisible by the perfect square $4$ ; Split radicand into factors |
| $=3\sqrt{4\cdot 2}$ | Apply product rule |
| $=3\sqrt{4}\cdot\sqrt{2}$ | Take the square root of $4$ |
| $=3\cdot 2\sqrt{2}$ | Multiply |
| $=6\sqrt{2}$ | Our Answer |

The previous example could have been done in fewer steps if we had noticed that $72=36\cdot 2$, where 36 is the largest perfect square factor of 72. Often the time it takes to discover the larger perfect square is more than it would take to simplify the radicand in several steps.

Variables are often part of the radicand as well. To simplify radical expressions involving variables, use the property below:

> **SIMPLIFYING $\sqrt{a^2}$**
>
> For any *nonnegative* real number $a$,
>
> $$\sqrt{a^2}=a$$

Note this property only holds if $a$ is *nonnegative*. For this reason, we will assume that all variables involved in a radical expression are *nonnegative*.

When simplifying with variables, variables with exponents that are divisible by 2 are perfect squares. For example, by the power of a power rule of exponents, $(x^4)^2=x^8$. So $x^8$ is a perfect square and $\sqrt{x^8}=\sqrt{(x^4)^2}=x^4$. A shortcut for taking the square roots of variables is to divide the exponent by 2. In our example, $\sqrt{x^8} = x^4$ because we divide the exponent 8 by 2 to get 4. When squaring, we multiply the exponent by 2, so when taking a square root, we divide the exponent by 2.

This process is shown in the following example.

**Example 5.** Simplify.

| | |
|---|---|
| $-5\sqrt{18x^4y^6z^{10}}$ | $18$ is divisible by the perfect square $9$; Split radicand into factors |
| $=-5\sqrt{9\cdot 2x^4y^6z^{10}}$ | Apply product rule |
| $=-5\sqrt{9}\cdot\sqrt{2}\cdot\sqrt{x^4}\cdot\sqrt{y^6}\cdot\sqrt{z^{10}}$ | Simplify roots; divide exponents by $2$ |
| $=-5\cdot 3x^2y^3z^5\sqrt{2}$ | Multiply coefficients |
| $=-15x^2y^3z^5\sqrt{2}$ | Our Answer |

We can't always evenly divide the exponent of a variable by 2. Sometimes we have a remainder. If there is a remainder, this means the variable with an exponent equal to the remainder will remain inside the radical sign. On the outside of the radical, the exponent of the variable will be equal to the whole number part. This process is shown in the following example.

**Example 6.** Simplify.

| | |
|---|---|
| $\sqrt{20x^5y^9z^6}$ | $20$ is divisible by the perfect square $4$; Split radicand into factors |
| $=\sqrt{4\cdot 5x^5y^9z^6}$ | Apply product rule |
| $=\sqrt{4}\cdot\sqrt{5}\cdot\sqrt{x^5}\cdot\sqrt{y^9}\cdot\sqrt{z^6}$ | Simplify roots; divide exponents by $2$, remainder is left inside |
| $=2x^2y^4z^3\sqrt{5xy}$ | Our Answer |

In the previous example, for the variable *x*, we divided $\frac{5}{2}=2\,R\,1$, so $x^2$ came out of the radicand and $x^1 = x$ remained inside the radicand. For the variable $y$, we divided $\frac{9}{2}=4\,R\,1$, so $y^4$ came out of the radicand and $y^1 = y$ remained inside. For the variable $z$, we divided $\frac{6}{2}=3\,R\,0$, so $z^3$ came out of the radicand and no $zs$ remained inside.

# Practice Exercises
## Section 3.1: Square Roots

**Simplify.** Assume that all variables represent nonnegative real numbers.

1) $\sqrt{36}$
2) $\sqrt{-100}$
3) $-\sqrt{196}$
4) $\sqrt{12}$
5) $\sqrt{125}$
6) $\sqrt{72}$
7) $\sqrt{245}$
8) $3\sqrt{24}$
9) $5\sqrt{48}$
10) $6\sqrt{128}$
11) $-8\sqrt{392}$
12) $-7\sqrt{63}$
13) $\sqrt{192n}$
14) $\sqrt{343b}$
15) $\sqrt{169v^2}$
16) $\sqrt{100n^3}$
17) $\sqrt{252x^2}$
18) $\sqrt{200a^3}$
19) $-\sqrt{100k^4}$
20) $-4\sqrt{175p^4}$
21) $-7\sqrt{64x^4}$
22) $-5\sqrt{36m}$
23) $\sqrt{45x^2y^2}$
24) $\sqrt{72a^3b^4}$
25) $\sqrt{16x^3y^3}$
26) $\sqrt{98a^4b^2}$
27) $\sqrt{320x^4y^4}$
28) $\sqrt{512m^4n^3}$
29) $6\sqrt{80xy^2}$
30) $8\sqrt{98mn}$
31) $5\sqrt{245x^2y^3}$
32) $2\sqrt{72x^2y^2}$
33) $-2\sqrt{180u^3v}$
34) $-5\sqrt{28x^3y^4}$
35) $-8\sqrt{108x^4y^2z^4}$
36) $6\sqrt{50a^4bc^2}$
37) $2\sqrt{80hj^4k}$
38) $-\sqrt{32xy^2z^3}$
39) $-4\sqrt{54mnp^2}$
40) $-8\sqrt{56m^2p^4q}$

# ANSWERS to Practice Exercises
## Section 3.1: Square Roots

1) $6$
2) not a real number
3) $-14$
4) $2\sqrt{3}$
5) $5\sqrt{5}$
6) $6\sqrt{2}$
7) $7\sqrt{5}$
8) $6\sqrt{6}$
9) $20\sqrt{3}$
10) $48\sqrt{2}$
11) $-112\sqrt{2}$
12) $-21\sqrt{7}$
13) $8\sqrt{3n}$
14) $7\sqrt{7b}$
15) $13v$
16) $10n\sqrt{n}$
17) $6x\sqrt{7}$
18) $10a\sqrt{2a}$
19) $-10k^2$
20) $-20p^2\sqrt{7}$
21) $-56x^2$
22) $-30\sqrt{m}$
23) $3xy\sqrt{5}$
24) $6ab^2\sqrt{2a}$
25) $4xy\sqrt{xy}$
26) $7a^2b\sqrt{2}$
27) $8x^2y^2\sqrt{5}$
28) $16m^2n\sqrt{2n}$
29) $24y\sqrt{5x}$
30) $56\sqrt{2mn}$
31) $35xy\sqrt{5y}$
32) $12xy\sqrt{2}$
33) $-12u\sqrt{5uv}$
34) $-10xy^2\sqrt{7x}$
35) $-48x^2yz^2\sqrt{3}$
36) $30a^2c\sqrt{2b}$
37) $8j^2\sqrt{5hk}$
38) $-4yz\sqrt{2xz}$
39) $-12p\sqrt{6mn}$
40) $-16mp^2\sqrt{14q}$