Section 3.5: Rationalize Denominators

Objective: Rationalize the denominators of radical expressions.

It is considered bad practice to have a radical in the denominator of a fraction in final form. If there is a radical in the denominator, we will *rationalize* it or clear out any radicals in the denominator.

RATIONALIZING DENOMINATORS CONTAINING ONE TERM

First, we will focus on rationalizing denominators with a single radical term that is a square root in the denominator. Multiply both the numerator and denominator by the same square root to produce a perfect square in the denominator. Use the property for a *nonnegative* number *a:* $(\sqrt{a})(\sqrt{a}) = (\sqrt{a^2}) = a$.

Example 1. Rationalize the denominator.

$\dfrac{\sqrt{6}}{\sqrt{5}}$	Multiply numerator and denominator by $\sqrt{5}$
$= \dfrac{\sqrt{6}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}}$	Multiply the numerators; multiply $\sqrt{5} \cdot \sqrt{5} = \sqrt{25} = 5$ in the denominator
$= \dfrac{\sqrt{30}}{5}$	Our Answer

Example 2. Rationalize the denominator.

$\dfrac{12}{\sqrt{3x}}$	Multiply numerator and denominator by $\sqrt{3x}$
$= \dfrac{12}{\sqrt{3x}} \cdot \dfrac{\sqrt{3x}}{\sqrt{3x}}$	Multiply the numerators; multiply $\sqrt{3x} \cdot \sqrt{3x} = \sqrt{9x^2} = 3x$ in the denominator
$= \dfrac{12\sqrt{3x}}{3x}$	Reduce the fraction
$= \dfrac{4\sqrt{3x}}{x}$	Our Answer

Example 3. Rationalize the denominator.

$\frac{2+\sqrt{3}}{\sqrt{7}}$ Multiply numerator and denominator by $\sqrt{7}$

$=\frac{(2+\sqrt{3})}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}$ Distribute in the numerator; multiply $\sqrt{7}\cdot\sqrt{7}=\sqrt{49}=7$ in the denominator

$=\frac{2\sqrt{7}+\sqrt{21}}{7}$ Our Answer

Example 4. Rationalize the denominator.

$\frac{\sqrt{3}-9}{2\sqrt{6}}$ Multiply numerator and denominator by $\sqrt{6}$

$=\frac{(\sqrt{3})-9}{2\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}$ Distribute in the numerator; multiply $\sqrt{6}\cdot\sqrt{6}=\sqrt{36}=6$ in the denominator

$=\frac{\sqrt{18}-9\sqrt{6}}{2\cdot 6}$ Simplify radicals in numerator*;* multiply denominator

$=\frac{\sqrt{9\cdot 2}-9\sqrt{6}}{12}$ Take square root where possible

$=\frac{3\sqrt{2}-9\sqrt{6}}{12}$ Factor numerator

$=\frac{3\left(\sqrt{2}-3\sqrt{6}\right)}{12}$ Reduce by dividing common factor of 3

$=\frac{\sqrt{2}-3\sqrt{6}}{4}$ Our Answer

It is important to remember that when reducing the fraction, we cannot reduce with just the 3 and 12 or just the 9 and 12. When we have addition or subtraction in the numerator or denominator, factor and then divide out any common factors.

As we rationalize denominators, it will always be important to constantly check our answer to see if it can be simplified more. We ask ourselves, can the fraction be reduced? Can the radicals be simplified? These steps may happen several times on our way to the solution.

RATIONALIZING DENOMINATORS CONTAINING TWO TERMS

Now we will focus on rationalizing denominators containing two terms with one or more square roots. We will use a different strategy to rationalize the denominator than we did when the denominator had one radical term.

Consider $\frac{2}{\sqrt{3}-5}$. If we were to multiply the denominator by $\sqrt{3}$, we would distribute and end up with $3-5\sqrt{3}$. We have not cleared the radical from the denominator so our current method will not work.

Instead, we will multiply numerator and denominator by the **conjugate** of the denominator. The conjugate has the same terms but with the opposite sign in the middle. In our example with $\sqrt{3}-5$ in the denominator, its conjugate is $\sqrt{3}+5$. When we multiply the conjugates, we get:

$$(\sqrt{3}-5)(\sqrt{3}+5)=3+5\sqrt{3}-5\sqrt{3}-25=3-25=-22$$

When multiplying conjugates, we will no longer have a radical in the denominator.

Example 5. Rationalize the denominator.

$\frac{2}{\sqrt{3}-5}$	Multiply numerator and denominator by $\sqrt{3}+5$, the conjugate of the denominator.
$=\frac{2}{\sqrt{3}-5}\left(\frac{\sqrt{3}+5}{\sqrt{3}+5}\right)$	Distribute in the numerator; multiply conjugates in the denominator.
$=\frac{2\sqrt{3}+10}{3+5\sqrt{3}-5\sqrt{3}-25}$	Simplify the denominator.
$=\frac{2\sqrt{3}+10}{-22}$	Factor numerator using a GCF of -2
$=\frac{-2\left(-\sqrt{3}-5\right)}{-22}$	Reduce by dividing common factor of -2
$=\frac{-\sqrt{3}-5}{11}$	Our Answer

Example 6. Rationalize the denominator.

$\dfrac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}$	Multiply numerator and denominator by $\sqrt{5}-\sqrt{3}$, the conjugate of the denominator.
$=\dfrac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}\left(\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)$	Distribute in the numerator; multiply conjugates in the denominator.
$=\dfrac{\sqrt{75}-\sqrt{45}}{5-\sqrt{15}+\sqrt{15}-3}$	Simplify the radicals in the numerator; simplify the denominator.
$=\dfrac{\sqrt{25}\sqrt{3}-\sqrt{9}\sqrt{5}}{2}$	Take square roots.
$=\dfrac{5\sqrt{3}-3\sqrt{5}}{2}$	Our Answer

Example 7. Rationalize the denominator.

$\dfrac{2\sqrt{3x}}{4-\sqrt{5x^3}}$	Multiply numerator and denominator by $4+\sqrt{5x^3}$, the conjugate of the denominator.
$=\dfrac{2\sqrt{3x}}{4-\sqrt{5x^3}}\left(\dfrac{4+\sqrt{5x^3}}{4+\sqrt{5x^3}}\right)$	Distribute in the numerator; multiply conjugates in the denominator.
$=\dfrac{8\sqrt{3x}+2\sqrt{15x^4}}{16+4\sqrt{5x^3}-4\sqrt{5x^3}-5x^3}$	Simplify the radicals in the numerator; simplify the denominator.
$=\dfrac{8\sqrt{3x}+2x^2\sqrt{15}}{16-5x^3}$	Our Answer

Example 8. Rationalize the denominator.

$\dfrac{3-\sqrt{5}}{2-\sqrt{3}}$	Multiply numerator and denominator by $2+\sqrt{3}$, the conjugate of the denominator.
$=\dfrac{3-\sqrt{5}}{2-\sqrt{3}}\left(\dfrac{2+\sqrt{3}}{2+\sqrt{3}}\right)$	Distribute in the numerator; multiply conjugates in the denominator.
$=\dfrac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4+2\sqrt{3}-2\sqrt{3}-3}$	Simplify the radicals in the numerator; simplify the denominator.
$=\dfrac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{1}$	Divide each term by 1.
$=6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}$	Our Answer

Practice Exercises
Section 3.5: Rationalize Denominators

Rationalize the denominator.

1) $\frac{1}{\sqrt{2}}$

2) $\frac{5}{\sqrt{3}}$

3) $\frac{\sqrt{10}}{\sqrt{6}}$

4) $\frac{\sqrt{2}}{3\sqrt{5}}$

5) $\frac{2\sqrt{4}}{3\sqrt{3}}$

6) $\frac{4\sqrt{3}}{\sqrt{15}}$

7) $\frac{\sqrt{5}}{2-\sqrt{3}}$

8) $\frac{7}{\sqrt{3}+8}$

9) $\frac{2}{\sqrt{7}+\sqrt{2}}$

10) $\frac{2\sqrt{3}}{9+3\sqrt{11}}$

11) $\frac{1+\sqrt{3}}{5+\sqrt{2}}$

12) $\frac{6\sqrt{3}}{\sqrt{5}-3}$

13) $\frac{8}{\sqrt{13}-\sqrt{7}}$

14) $\frac{4}{2\sqrt{7}-2}$

15) $\frac{5x^2}{4\sqrt{3x^2y^3}}$

16) $\frac{4}{5\sqrt{3xy^4}}$

17) $\frac{\sqrt{2p^2}}{\sqrt{3p}}$

18) $\frac{\sqrt{8n^2}}{\sqrt{10n}}$

19) $\frac{2-5\sqrt{5}}{4\sqrt{13}}$

20) $\frac{\sqrt{5}+4}{4\sqrt{17}}$

21) $\frac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$

22) $\frac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$

ANSWERS to Practice Exercises
Section 3.5: Rationalize Denominators

1) $\dfrac{\sqrt{2}}{2}$

2) $\dfrac{5\sqrt{3}}{3}$

3) $\dfrac{\sqrt{15}}{3}$

4) $\dfrac{\sqrt{10}}{15}$

5) $\dfrac{4\sqrt{3}}{9}$

6) $\dfrac{4\sqrt{5}}{5}$

7) $2\sqrt{5}+\sqrt{15}$

8) $\dfrac{56-7\sqrt{3}}{61}$

9) $\dfrac{2\left(\sqrt{7}-\sqrt{2}\right)}{5}$

10) $\dfrac{\sqrt{33}-3\sqrt{3}}{3}$

11) $\dfrac{5-\sqrt{2}+5\sqrt{3}-\sqrt{6}}{23}$

12) $\dfrac{-3\left(\sqrt{15}+3\sqrt{3}\right)}{2}$

13) $\dfrac{4\left(\sqrt{13}+\sqrt{7}\right)}{3}$

14) $\dfrac{1+\sqrt{7}}{3}$

15) $\dfrac{5x\sqrt{3y}}{12y^2}$

16) $\dfrac{4\sqrt{3x}}{15xy^2}$

17) $\dfrac{\sqrt{6p}}{3}$

18) $\dfrac{2\sqrt{5n}}{5}$

19) $\dfrac{2\sqrt{13}-5\sqrt{65}}{52}$

20) $\dfrac{\sqrt{85}+4\sqrt{17}}{68}$

21) $\dfrac{\sqrt{6}-9}{3}$

22) $\dfrac{\sqrt{30}-2\sqrt{3}}{18}$