# Section 4.2: Completing the Square

## Objective: Solve quadratic equations by completing the square.

In this section, we continue to address the question of how to solve any quadratic equation $ax^2 + bx + c = 0$. Now, we will learn a method known as **completing the square**. When completing the square, we will change the quadratic into a perfect square that can then be solved by applying the square root property. The next example reviews the square root property.

**Example 1**. Solve the equation.

| | |
|---|---|
| $(x+5)^2 = 18$ | Use square root property |
| $\sqrt{(x+5)^2} = \pm\sqrt{18}$ | Simplify radicals |
| $x+5 = \pm 3\sqrt{2}$ | Subtract 5 from both sides |
| $-5 = -5$ | |
| $x = -5 \pm 3\sqrt{2}$ | Our Solutions |

## COMPLETING THE SQUARE

To complete the square, or make our problem into the form of the previous example, we will be searching for the third term of a trinomial. If a quadratic is of the form $x^2 + bx + c$, and a perfect square, the third term, $c$, can be easily found by the formula $(\frac{1}{2} \cdot b)^2$. This is shown in the following examples, where we find the number that completes the square, and then factor that perfect square trinomial.

**Example 2**. Find the value of $c$ that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

| | |
|---|---|
| $x^2 + 8x + c$ | $c = \left(\frac{1}{2} \cdot b\right)^2$ and our $b = 8$ |
| | The third term that completes the square is 16: $\left(\frac{1}{2} \cdot 8\right)^2 = (4)^2 = 16$ |
| $x^2 + 8x + 16$ $= (x+4)(x+4)$ | Our expression is a perfect square; factor |
| $= (x+4)^2$ | Our Answer |

**Example 3**. Find the value of $c$ that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

$x^2 - 7x + c$ $\quad$ $c = \left(\frac{1}{2} \cdot b\right)^2$ and our $b = -7$

The third term that completes the square is $\frac{49}{4}$ :

$$\left(\frac{1}{2} \cdot -7\right)^2 = \left(-\frac{7}{2}\right)^2 = \frac{49}{4}$$

$x^2 - 7x + \frac{49}{4}$ $\quad$ Our expression is a perfect square; factor

$= \left(x - \frac{7}{2}\right)\left(x - \frac{7}{2}\right)$

$= \left(x - \frac{7}{2}\right)^2$ $\quad$ Our Answer

**Example 4**. Find the value of $c$ that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

$x^2 + \frac{5}{3}x + c$ $\quad$ $c = \left(\frac{1}{2} \cdot b\right)^2$ and our $b = \frac{5}{3}$

The third term that completes the square is $\frac{25}{36}$ :

$$\left(\frac{1}{2} \cdot \frac{5}{3}\right)^2 = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

$x^2 + \frac{5}{3}x + \frac{25}{36}$ $\quad$ Our expression is a perfect square; factor

$= \left(x + \frac{5}{6}\right)^2$ $\quad$ Our Answer

## SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

The process in the previous examples, combined with the square root property, is used to solve quadratic equations by completing the square. The following six steps describe the process used to solve a quadratic equation by completing the square, along with a practice example to demonstrate each step.

| **Steps for Solving by Completing the Square** | **Example** |
|---|---|
| | $3x^2+18x-6=0$ |
| 1. Separate the constant term from the variable terms. | $\begin{array}{r} +6=+6 \\ \hline 3x^2+18x \quad =6 \end{array}$ |
| 2. If $a \neq 1$, then divide each term by $a$. | $\frac{3}{3}x^2+\frac{18}{3}x = \frac{6}{3}$ <br> $x^2+6x = 2$ |
| 3. Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$. | $\left(\frac{1}{2}\cdot 6\right)^2=3^2=9$ |
| 4. Add the resulting value to both sides of the equation. | $\begin{array}{r} x^2+6x \quad =2 \\ +9=+9 \\ \hline x^2+6x+9=11 \end{array}$ |
| 5. Factor the perfect square trinomial. | $(x+3)^2=11$ |
| 6. Solve by applying the square root property. | $\sqrt{(x+3)^2}=\pm\sqrt{11}$ <br> $x+3=\pm\sqrt{11}$ <br> $\begin{array}{r} -3 \ =-3 \\ \hline x=-3\pm\sqrt{11} \end{array}$ |

The advantage of this method is that it can be used to solve **any** quadratic equation. The following examples show how completing the square can give us rational solutions, irrational solutions, and even complex solutions.

**Example 5**. Solve the equation by completing the square.

$$\begin{array}{r} 2x^2+20x+48=0 \\ -48=-48 \\ \hline \end{array}$$

Separate the constant term from variable terms
Subtract 48 from both sides of the equation

$$\frac{2x^2}{2}+\frac{20x}{2} = \frac{-48}{2}$$

Divide each term by 2

$$x^2+10x = -24$$

Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$

Our $b=10$; $\left(\frac{1}{2}\cdot 10\right)^2=(5)^2=25$

$$\begin{array}{r} x^2+10x \quad =-24 \\ +25=+25 \\ \hline \end{array}$$

Add 25 to both sides of the equation

$$x^2+10x+25=1$$

Factor the perfect square trinomial

$$(x+5)^2=1$$

Solve by applying the square root property

| | |
|---|---|
| $\sqrt{(x+5)^2} = \pm\sqrt{1}$ | Simplify radicals |
| $x+5=\pm 1$ | One equation for $+$, one equation for $-$ |
| $x+5=1$ or $x+5=-1$ | Subtract $5$ from both sides |
| $-5=-5$ $\quad$ $-5=-5$ | |
| $x=-4$ or $x=-6$ | Our Solutions |

**Example 6**. Solve the equation by completing the square.

| | |
|---|---|
| $x^2-3x-2=0$ | Separate the constant term from variable terms |
| $+2 = +2$ | Add 2 to both sides |
| $x^2-3x = 2$ | No need to divide since $a=1$ |
| | Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$ |
| | Our $b=-3$; $\left(\frac{1}{2}\cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ |
| | Add $\frac{9}{4}$ to both sides of the equation |
| $x^2-3x+\frac{9}{4}=2+\frac{9}{4}$ | Need common denominator (4) on right side of equation |
| | $\frac{2}{1}\left(\frac{4}{4}\right)+\frac{9}{4}=\frac{8}{4}+\frac{9}{4}=\frac{17}{4}$ |
| $x^2-3x+\frac{9}{4}=\frac{17}{4}$ | Factor the perfect square trinomial |
| $\left(x-\frac{3}{2}\right)^2=\frac{17}{4}$ | Solve by applying the square root property |
| $\sqrt{\left(x-\frac{3}{2}\right)^2}=\pm\sqrt{\frac{17}{4}}$ | Simplify radicals |
| $x-\frac{3}{2}=\pm\frac{\sqrt{17}}{2}$ | Add $\frac{3}{2}$ to both sides of the equation |
| $+\frac{3}{2} = +\frac{3}{2}$ | Notice that we already have a common denominator |
| $x=\frac{3\pm\sqrt{17}}{2}$ | Our Solutions |

As Example 6 has shown, when solving by completing the square, we will often need to use fractions and be comfortable finding common denominators and adding fractions together.

Sometimes when solving quadratic equations, the solutions are complex numbers, as is the case in Example 7.

**Example 7**. Solve the equation by completing the square.

| | |
|---|---|
| $x^2 - 6x + 30 = 0$ | Separate the constant term from variable terms |
| $\underline{\quad -30 = -30}$ | Subtract $30$ from both sides of the equation |
| $x^2 - 6x \quad = -30$ | No need to divide since $a = 1$ |
| | Find the value that completes the square: $\left(\frac{1}{2} \cdot b\right)^2$ |
| | Our $b = -6$; $\left(\frac{1}{2} \cdot -6\right)^2 = (-3)^2 = 9$ |
| $x^2 - 6x \quad = -30$ | Add $9$ to both sides of the equation |
| $\underline{\quad +9 = +9}$ | |
| $x^2 - 6x + 9 = -21$ | Factor the perfect square trinomial |
| $(x-3)^2 = -21$ | Solve by applying the square root property |
| $\sqrt{(x-3)^2} = \pm\sqrt{-21}$ | Simplify radicals |
| $x - 3 = \pm i\sqrt{21}$ | Add $3$ to both sides |
| $\underline{\quad +3 = +3}$ | |
| $x = 3 \pm i\sqrt{21}$ | Our Solutions |

We can use completing the square to solve any quadratic equation so we want to get comfortable using the six steps of this method.

# Practice Exercises
## Section 4.2: Completing the Square

**Find the value of $c$ that makes each expression a perfect square trinomial; then, factor that perfect square trinomial.**

1) $x^2 - 30x + c$
2) $a^2 + 24a + c$
3) $m^2 - 36m + c$
4) $x^2 + 34x + c$
5) $x^2 - 15x + c$
6) $r^2 + \frac{1}{9}r + c$
7) $y^2 + y + c$
8) $p^2 - 17p + c$

**Solve each equation by completing the square.**

9) $x^2 - 16x + 55 = 0$
10) $n^2 - 8n - 9 = 0$
11) $v^2 - 8v + 45 = 0$
12) $b^2 + 2b + 43 = 0$
13) $x^2 + 5x = 7$
14) $3k^2 + 2k - 4 = 0$
15) $-4z^2 + z + 1 = 0$
16) $8a^2 + 16a - 1 = 0$
17) $x^2 + 10x - 57 = 4$
18) $p^2 - 16p - 52 = 0$
19) $n^2 - 16n + 67 = 4$
20) $m^2 - 8m - 12 = 0$
21) $2x^2 + 4x + 38 = -6$
22) $6r^2 + 12r - 24 = -6$
23) $8b^2 + 16b - 37 = 5$
24) $6n^2 - 12n - 14 = 4$
25) $x^2 = -10x - 29$
26) $v^2 = 14v + 36$
27) $3k^2 + 9 = 6k$
28) $5n^2 = -10n + 15$
29) $p^2 - 8p = -55$
30) $x^2 + 8x + 15 = 8$
31) $7n^2 - n + 7 = 7n + 6n^2$
32) $n^2 + 4n = 12$
33) $8n^2 + 16n = 64$
34) $b^2 + 7b - 33 = 0$
35) $-5x^2 - 8x + 40 = -8$
36) $m^2 = -15 + 9m$
37) $4b^2 - 15b + 56 = 3b^2$
38) $10v^2 - 15v = 27 + 4v^2 - 6v$
39) $n^2 = -21 + 10n$
40) $a^2 - 56 = -10a$

# ANSWERS to Practice Exercises
## Section 4.2: Completing the Square

1) $225; (x-15)^2$
2) $144; (a+12)^2$
3) $324; (m-18)^2$
4) $289; (x+17)^2$
5) $\frac{225}{4}; \left(x-\frac{15}{2}\right)^2$
6) $\frac{1}{324}; \left(r+\frac{1}{18}\right)^2$
7) $\frac{1}{4}; \left(y+\frac{1}{2}\right)^2$
8) $\frac{289}{4}; \left(p-\frac{17}{2}\right)^2$

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9) $11, 5$
10) $9, -1$
11) $4+i\sqrt{29}, 4-i\sqrt{29}$
12) $-1+i\sqrt{42}, -1-i\sqrt{42}$
13) $\frac{-5+\sqrt{53}}{2}, \frac{-5-\sqrt{53}}{2}$
14) $\frac{-1+\sqrt{13}}{3}, \frac{-1-\sqrt{13}}{3}$
15) $\frac{1+\sqrt{17}}{8}, \frac{1-\sqrt{17}}{8}$
16) $\frac{-4+3\sqrt{2}}{4}, \frac{-4-3\sqrt{2}}{4}$
17) $-5+\sqrt{86}, -5-\sqrt{86}$
18) $8+2\sqrt{29}, 8-2\sqrt{29}$
19) $9, 7$
20) $4+2\sqrt{7}, 4-2\sqrt{7}$
21) $-1+i\sqrt{21}, -1-i\sqrt{21}$
22) $1, -3$
23) $\frac{3}{2}, -\frac{7}{2}$
24) $3, -1$
25) $-5+2i, -5-2i$
26) $7+\sqrt{85}, 7-\sqrt{85}$
27) $1+i\sqrt{2}, 1-i\sqrt{2}$
28) $1, -3$
29) $4+i\sqrt{39}, 4-i\sqrt{39}$
30) $-1, -7$
31) $7, 1$
32) $2, -6$
33) $2, -4$
34) $\frac{-7+\sqrt{181}}{2}, \frac{-7-\sqrt{181}}{2}$
35) $\frac{12}{5}, -4$
36) $\frac{9+\sqrt{21}}{2}, \frac{9-\sqrt{21}}{2}$
37) $8, 7$
38) $3, -\frac{3}{2}$
39) $7, 3$
40) $4, -14$