Chapter 4: Factoring Polynomials

Section 4.1: Factoring Using the Greatest Common Factor

Objective: Find the greatest common factor of a polynomial and factor it out of the expression.

The inverse of multiplying polynomials together is factoring polynomials. There are many benefits of a polynomial being factored. We use factored polynomials to help us solve equations, learn behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials, it is very important to have very strong factoring skills.

In this lesson, we will focus on factoring using the greatest common factor or GCF of a polynomial. When we multiplied polynomials, we multiplied monomials by polynomials by distributing, such as $4x^2(2x^2-3x+8)=8x^4-12x^3+32x^2$. In this lesson, we will work the same problem backwards. For example, we will start with $8x^4-12x^3+32x^2$ and try and work backwards to the $4x^2(2x^2-3x+8)$.

To do this, we have to be able to first identify what is the GCF of a polynomial. We will introduce this idea by looking at finding the GCF of several numbers. To find the GCF of several numbers, we are looking for the largest number that can divide each number without leaving a remainder. This can often be done with quick mental math. See the example below.

Example 1. Determine the greatest common factor.

Find the GCF of 15, 24, and 27

$\frac{15}{3}=5, \frac{24}{3}=8, \frac{27}{3}=9$ Each number can be divided by 3

$\text{GCF}=3$ Our Solution

When there are variables in our problem, we can first find the GCF of the numbers using mental math. Then, we take any variables that are in common with each term, using the lowest exponent. This is shown in the next example.

Example 2. Determine the greatest common factor.

Find the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$

$\frac{24}{6}=4, \frac{18}{6}=3, \frac{12}{6}=2$ Each number can be divided by 6.

x^2y x and y are in all 3; use lowest exponents

$$\text{GCF} = 6x^2y \quad \text{Our Solution}$$

To factor out a GCF from a polynomial, we first need to identify the GCF of all the terms. This is the part that goes in front of the parentheses. Then we divide each term by the GCF, and the quotients go inside the parentheses. This is shown in the following examples.

Example 3. Factor using the greatest common factor.

$$4x^2 - 20x + 16 \quad \text{GCF is 4; divide each term by 4}$$

$$\frac{4x^2}{4} = x^2, \frac{-20x}{4} = -5x, \frac{16}{4} = 4 \quad \text{Result is what is left in parentheses}$$

$$4(x^2 - 5x + 4) \quad \text{Our Solution}$$

With factoring, we can always check our solutions by multiplying (or distributing), and the product should be the original expression.

Example 4. Factor using the greatest common factor.

$$25x^4 - 15x^3 + 20x^2 \quad \text{GCF is } 5x^2\text{; divide each term by } 5x^2$$

$$\frac{25x^4}{5x^2} = 5x^2, \frac{-15x^3}{5x^2} = -3x, \frac{20x^2}{5x^2} = 4 \quad \text{Result is what is left in parentheses}$$

$$5x^2(5x^2 - 3x + 4) \quad \text{Our Solution}$$

Example 5. Factor using the greatest common factor.

$$3x^3y^2z + 5x^4y^3z^5 - 4xy^4 \quad \text{GCF is } xy^2\text{; divide each term by } xy^2$$

$$\frac{3x^3y^2z}{xy^2} = 3x^2z, \frac{-5x^4y^3z^5}{xy^2} = 5x^3yz^5, \frac{-4xy^4}{xy^2} = -4y^2 \quad \text{Result is what is left in parentheses}$$

$$xy^2(3x^2z + 5x^3yz^5 - 4y^2) \quad \text{Our Solution}$$

World View Note: The first recorded algorithm for finding the greatest common factor comes from the Greek mathematician Euclid around the year 300 BC!

Example 6. Factor using the greatest common factor.

$$21x^3 + 14x^2 + 7x \quad \text{GCF is } 7x\text{; divide each term by } 7x$$

$$\frac{21x^3}{7x} = 3x^2, \frac{14x^2}{7x} = 2x, \frac{7x}{7x} = 1 \quad \text{Result is what is left in parentheses}$$

$$7x(3x^2 + 2x + 1) \quad \text{Our Solution}$$

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was one of the terms, dividing gave an answer of 1. Students often try to factor out the $7x$ and get zero which is incorrect. Factoring will never make terms disappear. Anything divided by itself is 1, so be sure to not forget to put the 1 into the solution.

In the next example, we will factor out the negative of the GCF. Whenever the first term of a polynomial is negative, we will factor out the negative of the GCF.

Example 7. Factor using the negative of the greatest common factor.

$-12x^5y^2+6x^4y^4-8x^3y^5$	Negative of the GCF is $-2x^3y^2$; divide each term by $-2x^3y^2$
$\frac{-12x^5y^2}{-2x^3y^2}=6x^2, \frac{6x^4y^4}{-2x^3y^2}=-3xy^2, \frac{-8x^3y^5}{-2x^3y^2}=4y^3$	Result is what is left in parentheses
$-2x^3y^2(6x^2-3xy^2+4y^3)$	Our Solution

Often the second step is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parentheses as shown in the following two examples.

Example 8. Factor using the greatest common factor.

$18a^4b^3-27a^3b^3+9a^2b^3$	GCF is $9a^2b^3$; divide each term by $9a^2b^3$
$9a^2b^3(2a^2-3a+1)$	Our Solution

Again, in the previous example, when dividing $9a^2b^3$ by itself, the answer is 1, not zero. Be very careful that each term is accounted for in your final solution.

It is possible to have a problem where the GCF is 1. If the GCF is 1, then the polynomial cannot be factored. In this case, we state that the polynomial is **prime**. This is shown in the following example.

Example 9. Factor using the greatest common factor.

$8ab-17c+49$	GCF is 1 because there are no other factors in common to all 3 terms
$8ab-17c+49$	Our Solution: Prime

4.1 Practice

Factor each polynomial using the greatest common factor. If the first term of the polynomial is negative, then factor out the negative of the greatest common factor. If the expression cannot be factored, state that it is *prime*.

1) $9+8b^2$
2) $x-5$
3) $45x^2-25$
4) $-1-2n^2$
5) $56-35p$
6) $50x-80y$
7) $8ab-35a^2b$
8) $27x^2y^5-72x^3y^2$
9) $-3a^2b+6a^3b^2$
10) $8x^3y^2+4x^3$
11) $-5x^2-5x^3-15x^4$
12) $-32n^9+32n^6+40n^5$
13) $20x^4-30x+30$
14) $21p^6+30p^2+27$
15) $20x^4-30x+30$
16) $-10x^4+20x^2+12x$
17) $30b^9+5ab-15a^2$
18) $27y^7+12y^2x+9y^2$
19) $-48a^2b^2-56a^3b-56a^5b$
20) $30m^6+15mn^2-25$
21) $20x^8y^2z^2+15x^5y^2z+35x^3y^3z$
22) $3p+12q-15q^2r^2$
23) $50x^2y+10y^2+70xz^2$
24) $30y^4z^3x^5+50y^4z^5-10y^4z^3x$
25) $30qpr-5qp+5q$
26) $28b+14b^2+35b^3+7b^5$
27) $-18n^5+3n^3-21n+3$
28) $30a^8+6a^5+27a^3+21a^2$
29) $-40x^{11}-20x^{12}+50x^{13}-50x^{14}$
30) $-24x^6-4x^4+12x^3+4x^2$
31) $-32mn^8+4m^6n+12mn^4+16mn$

4.1 Answers

1) prime
2) prime
3) $5(9x^2 - 5)$
4) $-1\left(1+2n^2\right)$
5) $7(8-5p)$
6) $10(5x-8y)$
7) $ab(8-35a)$
8) $9x^2y^2(3y^3-8x)$
9) $-3a^2b(1-2ab)$
10) $4x^3(2y^2+1)$
11) $-5x^2(1+x+3x^2)$
12) $-8n^5(4n^4-4n-5)$
13) $10(2x^4-3x+3)$
14) $3(7p^6+10p^2+9)$
15) $4(7m^4+10m^3+2)$
16) $-2x(5x^3-10x-6)$
17) $5(6b^9+ab-3a^2)$
18) $3y^2(9y^5+4x+3)$
19) $-8a^2b(6b+7a+7a^3)$
20) $5(6m^6+3mn^2-5)$
21) $5x^3y^2z(4x^5z+3x^2+7y)$
22) $3(p+4q-5q^2r^2)$
23) $10(5x^2y+y^2+7xz^2)$
24) $10y^4z^3(3x^5+5z^2-x)$
25) $5q(6pr-p+1)$
26) $7b(4+2b+5b^2+b^4)$
27) $-3(6n^5-n^3+7n-1)$
28) $3a^2(10a^6+2a^3+9a+7)$
29) $-10x^{11}(4+2x-5x^2+5x^3)$
30) $-4x^2(6x^4+x^2-3x-1)$
31) $-4mn(8n^7-m^5-3n^3-4)$

Section 4.2: Factoring by Grouping

Objective: Factor polynomials with four terms using grouping.

The first thing we will always do, when factoring, is try to factor out a GCF. This GCF is often a monomial. For example, in the problem $5xy+10xz$, the GCF is the monomial $5x$; so, the factored expression is $5x(y+2z)$. However, a GCF does not have to be a monomial; it could be a binomial. To see this, consider the following two examples.

Example 1. Factor completely.

$3ax-7bx$	Both have x in common; factor out x
$x(3a-7b)$	Our Solution

In the next example, we have the same type of problem as in the example above; but, instead of x, the GCF is $(2a+5b)$.

Example 2. Factor completely.

$3a(2a+5b)-7b(2a+5b)$	Both have $(2a+5b)$ in common; factor out $(2a+5b)$
$(2a+5b)(3a-7b)$	Our Solution

In Example 2, we factored out a GCF that is a binomial, $(2a+5b)$. We will use this process of factoring a binomial GCF when the original polynomial has four terms and its GCF is 1.

When we need to factor a polynomial with four terms whose GCF is 1, we will have to use another strategy to factor. We will use a process known as **grouping**. We use grouping when factoring a polynomial with four terms. Remember, factoring is the reverse of multiplying, so first we will look at a multiplication problem and then try to reverse the process.

Example 3. Multiply.

$(2a+3)(5b+2)$	Distribute $(2a+3)$ into second parentheses
$5b(2a+3)+2(2a+3)$	Distribute each monomial
$10ab+15b+4a+6$	Our Solution

The product has four terms in it. We arrived at the solution by looking at the two parts, $5b(2a+3)$ and $2(2a+3)$. When we are factoring by grouping we will always divide the problem into two parts: the first two terms and the last two terms. Then, we can factor the GCF out of both the left and right groups. When we do this, our hope is what remains in the parentheses will match on both the left term and the right term. If they match, we can pull this matching GCF out front, putting the rest in parentheses, and the expression will be factored. The next example is the same problem worked backwards, factoring instead of multiplying.

Example 4. Factor completely.

$10ab+15b+4a+6$	Split expressions into two groups
$\boxed{10ab+15b}\boxed{+4a+6}$	GCF on left is $5b$; GCF on right is 2
$\boxed{5b(2a+3)}\boxed{+2(2a+3)}$	$(2a+3)$ is the same; factor out this GCF
$(2a+3)(5b+2)$	Our Solution

The key, for grouping to work, is after the GCF is factored out of the left and right groups, the two binomials must match exactly. If there is any difference between the two, we either have to do some adjusting or it cannot be factored using the grouping method. Consider the following example.

Example 5. Factor completely.

$6x^2+9xy-14x-21y$	Split expression into two groups
$\boxed{6x^2+9xy}\boxed{-14x-21y}$	GCF on left is $3x$; GCF on right is 7
$\boxed{3x(2x+3y)}\boxed{+7(-2x-3y)}$	The signs in the parentheses don't match!

When the signs don't match in both terms, we can easily make them match by factoring the negative of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second parentheses. In general, if the third term of the four - term expression is subtracted, then factor out the negative of the GCF for the second group of two terms.

$\boxed{3x(2x+3y)}\boxed{-7(2x+3y)}$	$(2x+3y)$ is the same; factor out this GCF
$(2x+3y)(3x-7)$	Our Solution

Often we can recognize early that we need to use the negative of the GCF when factoring. If the first term of the first binomial is positive in the problem, we will also want the first term of the second binomial to be positive. If the first term of the second binomial is negative, then we will use the negative of the GCF to be sure they match.

Example 6. Factor completely.

$5xy-8x-10y+16$	Split expression into two groups
$\boxed{5xy-8x}\boxed{-10y+16}$	GCF on left is x; GCF on right is -2
$\boxed{x(5y-8)}\boxed{-2(5y-8)}$	$(5y-8)$ is the same; factor out this GCF
$(5y-8)(x-2)$	Our Solution

Sometimes when factoring the GCF out of the left or right group there is no GCF to factor out other than one. In this case we will use either the GCF of 1 or -1. Often this is all we need to be sure the two binomials match.

Example 7. Factor completely.

$12ab-14a-6b+7$	Split expression into two groups
$\boxed{12ab-14a}\boxed{-6b+7}$	GCF on left is $2a$; negative of GCF on right is -1
$\boxed{2a(6b-7)}\boxed{-1(6b-7)}$	$(6b-7)$ is the same; factor out this GCF
$(6b-7)(2a-1)$	Our Solution

Example 8. Factor completely.

$6x^3-15x^2+2x-5$	Split expression into two groups
$\boxed{6x^3-15x^2}\boxed{+2x-5}$	GCF on left is $3x^2$; GCF on right is 1
$\boxed{3x^2(2x-5)}\boxed{+1(2x-5)}$	$(2x-5)$ is the same; factor out this GCF
$(2x-5)(3x^2+1)$	Our Solution

Example 9. Factor completely.

$12x^3+6x^2+5x+10$	Split expression into two groups
$\boxed{12x^3+6x^2}\boxed{+5x+10}$	GCF on left is $6x^2$; GCF on right is 5
$\boxed{6x^2(2x+1)}\boxed{+5(x+2)}$	Factors are not the same. This will happen even if we rearrange the terms. Therefore, it cannot be factored by grouping.
$12x^3+6x^2+5x+10$	Our Solution, Prime

4.2 Practice

Factor each expression completely.

1) $40r^3 - 8r^2 - 25r + 5$
2) $35x^3 - 10x^2 - 56x + 16$
3) $3n^3 - 2n^2 - 9n + 6$
4) $14v^3 + 10v^2 - 7v - 5$
5) $15b^3 + 21b^2 - 35b - 49$
6) $6x^3 - 48x^2 + 5x - 40$
7) $3x^3 + 15x^2 + 2x + 10$
8) $9x^3 + 3x^2 + 4x + 8$
9) $35x^3 - 28x^2 - 20x + 16$
10) $7n^3 + 21n^2 - 5n - 15$
11) $7xy - 49x + 5y - 35$
12) $42r^3 - 49r^2 + 18r - 21$
13) $32xy + 40x^2 + 12y + 15x$
14) $15ab - 6a + 5b^3 - 2b^2$
15) $16xy - 56x + 2y - 7$
16) $3mn - 8m + 15n - 40$
17) $x^3 - 5x^2 + 7x - 21$
18) $5mn + 2m - 25n - 10$
19) $40xy + 35x - 8y^2 - 7y$
20) $6a^2 + 3a - 4b^2 + 2b$
21) $32uv - 20u + 24v - 15$
22) $4uv + 14u^2 + 12v + 42u$
23) $10xy + 25x + 12y + 30$
24) $24xy - 20x - 30y^3 + 25y^2$
25) $3uv - 6u^2 - 7v + 14u$
26) $56ab - 49a - 16b + 14$
27) $2xy - 8x^2 + 7y^3 - 28y^2x$
28) $28p^3 + 21p^2 + 20p + 15$
29) $16xy - 6x^2 + 8y - 3x$
30) $8xy + 56x - y - 7$

4.2 Answers

1) $(8r^2-5)(5r-1)$
2) $(5x^2-8)(7x-2)$
3) $(n^2-3)(3n-2)$
4) $(2v^2-1)(7v+5)$
5) $(3b^2-7)(5b+7)$
6) $(6x^2+5)(x-8)$
7) $(3x^2+2)(x+5)$
8) Prime
9) $(7x^2-4)(5x-4)$
10) $(7n^2-5)(n+3)$
11) $(7x+5)(y-7)$
12) $(7r^2+3)(6r-7)$
13) $(8x+3)(4y+5x)$
14) $(3a+b^2)(5b-2)$
15) $(8x+1)(2y-7)$
16) $(m+5)(3n-8)$
17) Prime
18) $(m-5)(5n+2)$
19) $(5x-y)(8y+7)$
20) Prime
21) $(4u+3)(8v-5)$
22) $2(u+3)(2v+7u)$
23) $(5x+6)(2y+5)$
24) $(4x-5y^2)(6y-5)$
25) $(3u-7)(v-2u)$
26) $(7a-2)(8b-7)$
27) $(2x+7y^2)(y-4x)$
28) $(7p^2+5)(4p+3)$
29) $(2x+1)(8y-3x)$
30) $(8x-1)(y+7)$

Section 4.3: Factoring Trinomials When the Leading Coefficient is One

Objective: Factor trinomials when the leading coefficient or the coefficient of x^2 is 1.

Factoring polynomials with three terms, or factoring trinomials, is the most important type of factoring to be mastered. Since factoring can be thought of as the reverse of multiplication, we will start with a multiplication problem and look at how we can reverse the process.

Example 1. Multiply and simplify.

$(x+6)(x-4)$	Distribute $(x+6)$ through second parentheses
$x(x+6)-4(x+6)$	Distribute each monomial through parentheses
$x^2+6x-4x-24$	Combine like terms
$x^2+2x-24$	Our Solution

You may notice that if you reverse the last three steps, the process looks like grouping. This is because it is grouping! The GCF of the left two terms is x and the negative of the GCF of the second two terms is -4. The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This is shown in the following example, which is the same problem worked backwards:

Example 2. Factor completely.

$x^2+2x-24$	Split middle term into $+6x-4x$
$x^2+6x-4x-24$	Grouping; GCF on left is x; negative of GCF on right is -4
$x(x+6)-4(x+6)$	$(x+6)$ is the same; factor out this GCF
$(x+6)(x-4)$	Our Solution

The trick to making these problems work is in how we split the middle term. Why did we pick $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that works! So, how do we know what is the one combination that works? To find the correct way to split the middle term, we find a pair of numbers that multiply to obtain the last term in the trinomial and also sum to the number that is the coefficient of the middle term of the trinomial. In the previous example that would mean we wanted to multiply to -24 and sum to 2. The only numbers that can do this are 6 and -4 ($6\cdot -4=-24$ and $6+(-4)=2$). This process is shown in the next few examples.

Example 3. Factor completely.

$x^2+9x+18$	Multiply to 18; sum to 9
$x^2+6x+3x+18$	6 and 3; split the middle term
$x(x+6)+3(x+6)$	Factor by grouping
$(x+6)(x+3)$	Our Solution

Example 4. Factor completely.

x^2-4x+3	Multiply to 3; sum to -4
$x^2-3x-x+3$	-3 and -1; split the middle term
$x(x-3)-1(x-3)$	Factor by grouping
$(x-3)(x-1)$	Our Solution

Example 5. Factor completely.

$x^2-8x-20$	Multiply to -20; sum to -8
$x^2-10x+2x-20$	-10 and 2; split the middle term
$x(x-10)+2(x-10)$	Factor by grouping
$(x-10)(x+2)$	Our Solution

Often when factoring, we have two variables. These problems solve just like problems with one variable, using the coefficients to decide how to split the middle term.

Example 6. Factor completely.

$a^2-9ab+14b^2$	Multiply to 14; sum to -9
$a^2-7ab-2ab+14b^2$	-7 and -2; split the middle term
$a(a-7b)-2b(a-7b)$	Factor by grouping
$(a-7b)(a-2b)$	Our Solution

Warning! Notice that it is very important to be aware of negatives, as we find the pair of numbers we will use to split the middle term. Consider the following example, done incorrectly, ignoring negative signs:

Factor x^2+5x-6	Multiply to 6; sum to 5
$x^2+2x+3x-6$	2 and 3; split the middle term
$x(x+2)+3(x-2)$	Factor by grouping
???	Binomials do not match!

Because we did not use the negative sign with the 6 to find our pair of numbers, the binomials did not match and grouping was not able to work at the end. Now the problem will be done correctly:

Example 7. Factor completely.

x^2+5x-6	Multiply to -6; sum to 5
$x^2+6x-1x-6$	6 and -1; split the middle term
$x(x+6)-1(x+6)$	Factor by grouping
$(x+6)(x-1)$	Our Solution

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1; our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it, however, when we have no number (technically we have a 1) in front of x^2. In all of the problems we have factored in this lesson, there is no number written in front of x^2. If this is the case, then we can use this shortcut. This is shown in the next few examples.

Example 8. Factor completely.

$x^2-7x-18$	Multiply to -18; sum to -7
	-9 and 2; write the factors
$(x-9)(x+2)$	Our Solution

Example 9. Factor completely.

$m^2-mn-30n^2$	Multiply to -30; sum to -1
	5 and -6; write the factors; don't forget the second variable
$(m+5n)(m-6n)$	Our Solution

It is possible to have a problem that does not factor. If there is no combination of numbers that multiplies and adds up to the correct numbers, then we say we cannot factor the polynomial or we say the polynomial is prime. This is shown in the following example.

Example 10. Factor completely.

x^2+2x+6	Multiply to 6; sum to 2
$1\cdot 6$ and $2\cdot 3$	Only possibilities to multiply to 6; none sum to 2
Prime	Our Solution

When factoring, it is important not to forget about the GCF. If all of the terms in a problem have a common factor, we will want to first factor out the GCF before we attempt using any other method. The next three examples illustrate this technique:

Example 11. Factor completely.

$3x^2-24x+45$	GCF of all terms is 3; factor out 3
$3\left(x^2-8x+15\right)$	Multiply to 15; sum to -8
	-5 and -3; write the factors
$3\left(x^2-8x+15\right)$	
$3(x-5)(x-3)$	Our Solution

Example 12. Factor completely.

$4x^2y-8xy-32y$	GCF of all terms is $4y$; factor out $4y$
$4y(x^2-2x-8)$	Multiply to -8; sum to -2
	-4 and 2; write the factors
$4y(x-4)(x+2)$	Our Solution

Example 13. Factor completely.

$7a^4b^2+28a^3b^2-35a^2b^2$	GCF of all terms is $7a^2b^2$; factor out $7a^2b^2$
$7a^2b^2(a^2+4a-5)$	Multiply to -5; sum to 4
	-1 and 5; write the factors
$7a^2b^2(a-1)(a+5)$	Our Solution

Again it is important to comment on the shortcut of jumping right to the factors. This only works if there is no written coefficient of x^2; that is, the leading coefficient is understood to be 1. Be careful not to use this shortcut on all factoring problems!

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables we are solving for, just as codes used letters to represent an unknown message.

4.3 Practice

Factor each expression completely.

1) $p^2+17p+72$
2) x^2+x-72
3) n^2-9n+8
4) x^2+x-30
5) $x^2-9x-10$
6) $x^2+13x+40$
7) $b^2+12b+32$
8) $b^2-17b+70$
9) $x^2+3x-70$
10) $x^2+3x-18$
11) $n^2-8n+15$
12) $a^2-6a-27$
13) $p^2+15p+54$
14) $p^2+7p-30$
15) c^2-4c+9
16) $m^2-15mn+50n^2$
17) $u^2-8uv+15v^2$
18) $m^2-3mn-40n^2$
19) $m^2+2mn-8n^2$
20) $x^2+10xy+16y^2$
21) $x^2-11xy+18y^2$
22) $u^2-9uv+14v^2$
23) $x^2+xy-12y^2$
24) $x^2+14xy+45y^2$
25) $x^2+4xy-12y^2$
26) $4x^2+52x+168$
27) $5a^2+60a+100$
28) $7w^2+5w-35$
29) $6a^2+24a-192$
30) $5v^2+20v-25$
31) $6x^2+18xy+12y^2$

32) $5m^2 + 30mn - 80n^2$

33) $6x^2 + 96xy + 378y^2$

34) $6m^2 - 36mn - 162n^2$

35) $n^2 - 15n + 56$

36) $5n^2 - 45n + 40$

4.3 Answers

1) $(p+9)(p+8)$
2) $(x-8)(x+9)$
3) $(n-8)(n-1)$
4) $(x-5)(x+6)$
5) $(x+1)(x-10)$
6) $(x+5)(x+8)$
7) $(b+8)(b+4)$
8) $(b-10)(b-7)$
9) $(x-7)(x+10)$
10) $(x-3)(x+6)$
11) $(n-5)(n-3)$
12) $(a+3)(a-9)$
13) $(p+6)(p+9)$
14) $(p+10)(p-3)$
15) Prime
16) $(m-5n)(m-10n)$
17) $(u-5v)(u-3v)$
18) $(m+5n)(m-8n)$
19) $(m+4n)(m-2n)$
20) $(x+8y)(x+2y)$
21) $(x-9y)(x-2y)$
22) $(u-7v)(u-2v)$
23) $(x-3y)(x+4y)$
24) $(x+5y)(x+9y)$
25) $(x+6y)(x-2y)$
26) $4(x+7)(x+6)$
27) $5(a+10)(a+2)$
28) Prime
29) $6(a-4)(a+8)$
30) $5(v-1)(v+5)$
31) $6(x+2y)(x+y)$
32) $5(m-2n)(m+8n)$
33) $6(x+9y)(x+7y)$
34) $6(m-9n)(m+3n)$
35) $(n-8)(n-7)$

36) $5(n-8)(n-1)$

Section 4.4: Factoring Special Forms of Polynomials

Objective: Identify and factor special forms of polynomials including a difference of two squares and perfect square trinomials.

When factoring, there are a few special forms of polynomials that, if we can recognize them, help us factor polynomials. The first is one we have seen before. When multiplying special products, we found that a sum and a difference could multiply to a difference of two squares. Here, we will use this special product to help us factor the difference of two squares.

Difference of Two Squares: $a^2 - b^2 = (a+b)(a-b)$

If we are subtracting two squares, then the expression will always factor to the sum and difference of the square roots.

Example 1. Factor completely.

x^2-16	Subtracting two squares; the square roots are x and 4
$(x+4)(x-4)$	Our Solution

Example 2. Factor completely.

$9a^2-25b^2$	Subtracting two squares; the square roots are $3a$ and $5b$
$(3a+5b)(3a-5b)$	Our Solution

In the next example, we will see that, generally, the sum of two squares cannot be factored.

Example 3. Factor completely.

x^2+36	No bx term; we use $0x$.
$x^2+0x+36$	Multiply to 36; sum to 0
$1\cdot 36,\ 2\cdot 18,\ 3\cdot 12,\ 4\cdot 9,\ 6\cdot 6$	No combinations that multiplies to 36 and sum to 0
Prime (cannot be factored)	Our Solution

It turns out that a sum of two squares is generally considered to be prime when the exponent is 2. If the exponent is greater than 2, then factoring the sum of two squares will go beyond the scope of this course.

Sum of Two Squares: $a^2+b^2=$ **prime (generally cannot be factored)**

A great example where we see a sum of two squares and a difference of two squares together would be factoring a difference of fourth powers. Because the square root of a fourth power is a square $\left(\sqrt{a^4}=a^2\right)$, we can factor a difference of fourth powers, just like we factor a difference of two squares, to a sum and difference of the square roots. This will give us two factors: one which will be a prime sum of two squares; and a second that will be a difference of two squares, which we can factor again. This is shown in the following two examples.

Example 4. Factor completely.

a^4-b^4	Difference of two squares with square roots a^2 and b^2
$(a^2+b^2)(a^2-b^2)$	The first factor is prime; the second is a difference of two squares with square roots a and b
$(a^2+b^2)(a+b)(a-b)$	Our Solution

Example 5. Factor completely.

x^4-16	Difference of two squares with square roots x^2 and 4
$(x^2+4)(x^2-4)$	The first factor is prime; the second is a difference of two squares with square roots x and 2
$(x^2+4)(x+2)(x-2)$	Our Solution

Another factoring formula is the perfect square trinomial. We had a shortcut for squaring binomials, which can be reversed to help us factor a perfect square trinomial.

Perfect Square Trinomial: $\boldsymbol{a^2+2ab+b^2=(a+b)^2}$

Here is how to recognize a perfect square trinomial:

(1) The first term is a square of a monomial or an integer.
(2) The middle term is two times the product of the square root of the first and last terms.
(3) The third term is a square of a monomial or an integer.

Then, we can factor a perfect square trinomial using the square roots of the first and last terms and the sign from the middle term. This is shown in the following examples.

Example 6. Factor completely.

x^2-6x+9	$x^2=(x)^2$; $6x=2(x)(3)$; $9=(3)^2$
$(x)^2-2\cdot x\cdot 3+(3)^2$	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$(x-3)^2$	Our Solution

Example 7. Factor completely.

$4x^2+20xy+25y^2$	$4x^2=(2x)^2$; $20xy=2(2x)(5y)$; $25y^2=(5y)^2$
$(2x)^2+2\cdot 2x\cdot 5y+(5y)^2$	Perfect square trinomial; use square roots from first and last terms and sign from the middle

$(2x+5y)^2$ Our Solution

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. For example, problems would be written as: “three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou.” If g represents a sheaf of good crop, m represents a sheaf of mediocre crop, and b respresents a sheaf of bad crop, then we can say that the polynomial $3g+2m+b$ equals 29.

The following table sumarizes all of the formulas that we can use to factor special forms of polynomials.

Factoring Special Forms of Polynomials

Difference of Two Squares:	$a^2-b^2=(a+b)(a-b)$
Sum of Two Squares:	$a^2+b^2=$ Prime (generally cannot be factored)
Perfect Square Trinomial:	$a^2+2ab+b^2=(a+b)^2$

As always, when factoring special forms of polynomials, it is important to check for a GCF first. Only after checking for a GCF should we use the special products used in the factoring formulas. This is shown in the following examples.

Example 8. Factor completely.

$72x^2-2$ GCF is 2

$2(36x^2-1)$ Difference of two squares; square roots are $6x$ and 1

$2(6x+1)(6x-1)$ Our Solution

Example 9. Factor completely.

$48x^2y-24xy+3y$ GCF is $3y$

$3y(16x^2-8x+1)$ $16x^2=(4x)^2; 8x=2(4x)(1); 1=(1)^2$; perfect square trinomial

$16x^2-8x+1=(4x)^2-2\cdot 4x\cdot 1+(1)^2$

Use square roots from first and last terms and sign from the middle

$3y(4x-1)^2$ Our Solution

4.4 Practice

Factor each expression completely.

1) $r^2 - 16$
2) $x^2 - 9$
3) $v^2 - 25$
4) $x^2 - 1$
5) $p^2 - 4$
6) $4v^2 - 1$
7) $9k^2 - 4$
8) $9a^2 - 1$
9) $3x^2 - 27$
10) $5n^2 - 20$
11) $16x^2 - 36$
12) $125x^2 + 45y^2$
13) $18a^2 - 50b^2$
14) $4m^2 + 64n^2$
15) $a^2 - 2a + 1$
16) $k^2 + 4k + 4$
17) $x^2 + 6x + 9$
18) $n^2 - 8n + 16$
19) $x^2 - 6x + 9$
20) $k^2 - 4k + 4$
21) $25p^2 - 10p + 1$
22) $x^2 + 2x + 1$
23) $25a^2 + 30ab + 9b^2$
24) $x^2 + 8xy + 16y^2$
25) $4a^2 - 20ab + 25b^2$
26) $49x^2 + 36y^2$
27) $8x^2 - 24xy + 18y^2$
28) $20x^2 + 20xy + 5y^2$
29) $a^4 - 81$
30) $x^4 - 256$
31) $16 - z^4$
32) $n^4 - 1$

33) $x^4 - y^4$

34) $16a^4 - b^4$

35) $m^4 - 81b^4$

36) $81c^4 - 16d^4$

37) $18m^2 - 24mn + 8n^2$

38) $w^4 + 225$

4.4 Answers

1) $(r+4)(r-4)$
2) $(x+3)(x-3)$
3) $(v+5)(v-5)$
4) $(x+1)(x-1)$
5) $(p+2)(p-2)$
6) $(2v+1)(2v-1)$
7) $(3k+2)(3k-2)$
8) $(3a+1)(3a-1)$
9) $3(x+3)(x-3)$
10) $5(n+2)(n-2)$
11) $4(2x+3)(2x-3)$
12) $5(25x^2+9y^2)$
13) $2(3a+5b)(3a-5b)$
14) $4(m^2+16n^2)$
15) $(a-1)^2$
16) $(k+2)^2$
17) $(x+3)^2$
18) $(n-4)^2$
19) $(x-3)^2$
20) $(k-2)^2$
21) $(5p-1)^2$
22) $(x+1)^2$
23) $(5a+3b)^2$
24) $(x+4y)^2$
25) $(2a-5b)^2$
26) Prime
27) $2(2x-3y)^2$
28) $5(2x+y)^2$
29) $(a^2+9)(a+3)(a-3)$
30) $(x^2+16)(x+4)(x-4)$
31) $(4+z^2)(2+z)(2-z)$
32) $(n^2+1)(n+1)(n-1)$
33) $(x^2+y^2)(x+y)(x-y)$

34) $(4a^2+b^2)(2a+b)(2a-b)$

35) $(m^2+9b^2)(m+3b)(m-3b)$

36) $(9c^2+4d^2)(3c+2d)(3c-2d)$

37) $2(3m-2n)^2$

38) Prime

Section 4.5: A General Factoring Strategy

Objective: Identify and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which strategy to use when. Here, we will attempt to organize all the different factoring methods we have seen. A large part of deciding how to solve a problem is based on how many terms are in the problem. For all problem types, we will always try to factor out the GCF first.

Factoring Strategy (GCF First!!!!!)

- **2 terms**: sum or difference of two squares:
  $a^2-b^2=(a+b)(a-b)$
  $a^2+b^2=$ prime (generally cannot be factored)
- **3 terms**: Watch for trinomials with leading coefficient of one and perfect square trinomials!
  $a^2+2ab+b^2=(a+b)^2$
- **4 terms**: grouping

We will use the above strategy to factor each of the following examples. Here, the emphasis will be on which strategy to use rather than the steps used in that method.

Example 1. Factor completely.

$x^2-23x+42$	GCF=1; so nothing to factor out of all three terms
	Three terms; multiply to 42; sum to -23
	-2 and -21; write the factors
$(x-2)(x-21)$	Our Solution

Example 2. Factor completely.

z^2+6z-9	GCF=1; so nothing to factor out of all three terms
	Three terms; multiply to -9; sum to 6
	Factors of -9: $(-1)(9), (1)(-9), (3)(-3)$; none sum to 6
Prime (cannot be factored)	Our Solution

Example 3. Factor completely.

$4x^2+56xy+196y^2$	GCF first; factor out 4 from each term
$4(x^2+14xy+49y^2)$	Three terms, $x^2=(x)^2; 14xy=2(x)(7y); 49y^2=(7y)^2$
	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$4(x+7y)^2$	Our Solution

Example 4. Factor completely.

$5x^2y+15xy-35x^2-105x$	GCF first; factor out $5x$ from each term
$5x(xy+3y-7x-21)$	Four terms; try grouping
$5x[y(x+3)-7(x+3)]$	$(x+3)$ match
$5x(x+3)(y-7)$	Our Solution

Example 5. Factor completely.

$100x^2-400$	GCF first; factor out 100 from each term
$100(x^2-4)$	Two terms; difference of two squares
$100(x+2)(x-2)$	Our Solution

Example 6. Factor completely.

$108x^3y^2-36x^2y^2+3xy^2$	GCF first; factor out $3xy^2$ from each term
$3xy^2(36x^2-12x+1)$	Three terms; $36x^2=(6x)^2; 12x=2(6x)(1); 1=(1)^2$
	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$3xy^2(6x-1)^2$	Our Solution

World View Note: Variables originated in ancient Greece where Aristotle would use a single capital letter to represent a number.

It is important to be comfortable and con dent not just with using all the factoring methods, but also with deciding on which method to use. This is why your practice with these problems is very important!

4.5 Practice

Factor each expression completely.

1) $16x^2+48xy+36y^2$
2) n^2-n
3) $x^2-4xy+3y^2$
4) $45u^2-150uv+125v^2$
5) $64x^2+49y^2$
6) m^2-4n^2
7) $3m^3-6m^2n-24n^2m$
8) $2x^3+6x^2y-20y^2x$
9) n^3+7n^2+10n
10) $16a^2-9b^2$
11) $5x^2+2x$
12) $2x^2-10x+12$
13) $3k^3-27k^2+60k$
14) $32x^2-18y^2$
15) $16x^2-8xy+y^2$
16) v^2+v
17) $27m^2-48n^2$
18) x^3+4x^2
19) $9n^3-3n^2$
20) $2m^2+6mn-20n^2$
21) $16x^2+1$
22) $9x^2-25y^2$
23) $mn+3m-4xn-12x$
24) $24az-18ah+60yz-45yh$
25) $20uv-60u^3-5xv+15xu^2$
26) $36b^2c-24b^2d+24xc-16xd$

4.5 Answers

1) $4(2x+3y)^2$
2) $n(n-1)$
3) $(x-3y)(x-y)$
4) $5(3u-5v)^2$
5) Prime
6) $(m+2n)(m-2n)$
7) $3m(m+2n)(m-4n)$
8) $2x(x+5y)(x-2y)$
9) $n(n+2)(n+5)$
10) $(4a+3b)(4a-3b)$
11) $x(5x+2)$
12) $2(x-2)(x-3)$
13) $3k(k-5)(k-4)$
14) $2(4x+3y)(4x-3y)$
15) $(4x-y)^2$
16) $v(v+1)$
17) $3(3m+4n)(3m-4n)$
18) $x^2(x+4)$
19) $3n^2(3n-1)$
20) $2(m-2n)(m+5n)$
21) Prime
22) $(3x+5y)(3x-5y)$
23) $(m-4x)(n+3)$
24) $3(2a+5y)(4z-3h)$
25) $5(4u-x)(v-3u^2)$
26) $4(3b^2+2x)(3c-2d)$

Section 4.6: Solving Equations by Factoring

Objective: Solve quadratic equations by factoring and using the zero product rule.

When solving linear equations such as $2x-5=21$; we can solve for the variable directly by adding 5 and dividing by 2, on both sides, to get 13. However, when we have x^2 (or a higher power of x), we cannot just isolate the variable, as we did with the linear equations. One method that we can use to solve for the variable is known as the zero product rule.

Zero Product Rule: If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

The zero product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero. We can use this to help us solve factored polynomials as in the following example.

Example 1. Solve the equation.

$$(2x-3)(5x+1)=0 \quad \text{One factor must be zero}$$

$$2x-3=0 \quad \text{or} \quad 5x+1=0 \quad \text{Set each factor equal to zero}$$

$$\begin{array}{rcl} \underline{+3 \quad +3} & & \underline{-1 \quad -1} \\ \frac{2x}{2}=\frac{3}{2} & \text{or} & \frac{5x}{5}=\frac{-1}{5} \end{array} \quad \text{Solve each equation}$$

$$x=\frac{3}{2} \quad \text{or} \quad \frac{-1}{5} \quad \text{Our Solution}$$

For the zero product rule to work, we must have factors to set equal to zero. This means if the problem is not already factored, we will need to factor it first, if at all possible.

Example 2. Solve the equation by factoring.

$$x^2-7x+12=0 \quad \text{Multiply to 12; sum to } -7$$

$$(x-3)(x-4)=0 \quad \text{Numbers are } -3 \text{ and } -4.$$

$$x-3=0 \quad \text{or} \quad x-4=0 \quad \text{Since one factor must be zero, set each factor equal to zero}$$

$$\underline{+3 \quad +3} \qquad \underline{+4 \quad +4} \quad \text{Solve each equation}$$

$$x=3 \quad \text{or} \quad x=4 \quad \text{Our Solution}$$

Another important part of the zero product rule is that before we factor, one side of the equation must be zero. If one side of the equation is not zero, we must move terms around so that one side of the equation is zero. Generally, we like the x^2 term to be positive.

Example 3. Solve the equation by factoring.

$x^2 = 8x - 15$	Set equal to zero by moving terms to the left
$\underline{-8x+15} \quad \underline{-8x+15}$	
$x^2 - 8x + 15 = 0$	Factor; multiply to 15; sum to -8
$(x-5)(x-3) = 0$	Numbers are -5 and -3
$x - 5 = 0$ or $x - 3 = 0$	Set each factor equal to zero
$\underline{+5 \quad +5} \quad \underline{+3 \quad +3}$	Solve each equation
$x = 5$ or $x = 3$	Our Solutions

Example 4. Solve the equation by factoring.

$(x-7)(x+3) = -9$	Not equal to zero; multiply first; use FOIL
$x^2 - 7x + 3x - 21 = -9$	Combine like terms
$x^2 - 4x - 21 = -9$	Move -9 to other side so equation equals zero
$\underline{+9 \quad +9}$	
$x^2 - 4x - 12 = 0$	Factor; multiply to -12; sum to -4
$(x-6)(x+2) = 0$	Numbers are -6 and $+2$
$x - 6 = 0$ or $x + 2 = 0$	Set each factor equal to zero
$\underline{+6 \quad +6} \quad \underline{-2 \quad -2}$	Solve each equation
$x = 6$ or $x = -2$	Our Solution

Example 5. Solve the equation by factoring.

$3x^2 + 4x - 5 = 7x^2 + 4x - 14$	Set equal to zero by moving terms to the right side of the equal sign
$\underline{-3x^2 - 4x + 5} \quad \underline{-3x^2 - 4x + 5}$	
$0 = 4x^2 - 9$	Factor using the difference of two squares
$0 = (2x+3)(2x-3)$	One factor must be zero
$2x + 3 = 0$ or $2x - 3 = 0$	Set each factor equal to zero
$\underline{-3 \quad -3} \quad \underline{+3 \quad +3}$	Solve each equation
$\frac{2x}{2} = \frac{-3}{2}$ or $\frac{2x}{2} = \frac{3}{2}$	
$x = -\frac{3}{2}$ or $\frac{3}{2}$	Our Solutions

Most problems with x^2 will have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 6. Solve the equation by factoring.

$4x^2 = 12x - 9$	Set equal to zero by moving terms to the left
$\underline{-12x+9 \quad -12x+9}$	
$4x^2 - 12x + 9 = 0$	Factor; $4x^2 = (2x)^2; 12x = 2(2x)(3); 9 = (3)^2$
$(2x-3)^2 = 0$	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$2x - 3 = 0$	Set this factor equal to zero
$\underline{+3 \quad +3}$	Solve the equation
$\frac{2x}{2} = \frac{3}{2}$	
$x = \frac{3}{2}$	Our Solution

As always, it will be important to factor out the GCF first, if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This may give us more than just two solutions. The next few examples illustrate this.

Example 7. Solve the equation by factoring.

$4x^2 = 8x$	Set equal to zero by moving the $8x$ to the left side of the equal sign
$\underline{-8x \quad -8x}$	
$4x^2 - 8x = 0$	
$4x(x-2) = 0$	One factor must be zero
$\frac{4x}{4} = \frac{0}{4}$ or $x - 2 = 0$	Set each factor equal to zero
$\underline{+2 \quad +2}$	Solve each equation
$x = 0$ or 2	Our Solution

Example 8. Solve the equation by factoring.

$2x^3 - 14x^2 + 24x = 0$	Factor out the GCF of $2x$
$2x(x^2 - 7x + 12) = 0$	Multiply to 12; sum to -7
$2x(x-3)(x+4) = 0$	Numbers are -3 and -4
$\frac{2x}{2} = \frac{0}{2}$ or $x - 3 = 0$ or $x - 4 = 0$	Set each factor equal to zero
$\underline{+3 \quad +3}$ $\underline{+4 \quad +4}$	Solve each equation
$x = 0$ or 3 or 4	Our Solutions

Example 9. Solve the equation by factoring.

$$3x^2+3x-60=0 \quad \text{Factor out the GCF of 3}$$
$$3(x^2+x-20)=0 \quad \text{Multiply to } -20\text{; sum to 1}$$
$$3(x+5)(x-4)=0 \quad \text{Numbers are 5 and } -4$$
$$3(x+5)\,(x-4) \quad \text{One factor must be zero}$$
$$3=0 \quad \text{or} \quad x+5=0 \quad \text{or} \quad x-4=0 \quad \text{Set each factor equal to zero}$$
$$3\neq 0 \qquad \underline{-5\ \ -5} \qquad \underline{+4\ \ +4} \quad \text{Solve each equation}$$
$$x=-5 \quad \text{or} \quad x=4$$
$$x=-5 \quad \text{or} \quad x=4 \quad \text{Our Solutions}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we get a false statement. No solution comes from this factor. Often a student will skip setting the GCF factor equal to zero if there are no variables in the GCF, which is acceptable.

Just as not all polynomials can be factored, not all equations can be solved by factoring. If an equation does not factor we will have to solve it using another method. These other methods are saved for another course.

World View Note: While factoring works great to solve problems with x^2, Tartaglia, in 16th century Italy, developed a method to solve problems with x^3. He kept his method a secret until another mathematician, Cardano, talked him out of his secret and published the results. To this day the formula is known as Cardano's Formula.

A question often asked is if it is possible to get rid of the square on the variable by taking the square root of both sides. While it is possible, there are a few properties of square roots that will not be covered in this course, and thus it is common to break a rule of roots that we are not aware of at this point. The short reason we want to avoid this for now is because taking a square root will only allow us to consider the principal or positive square root. For now, **never** take the square root of both sides!

4.6 Practice

Solve each equation by factoring.

1) $(k-7)(k+2)=0$
2) $(a+4)(a-3)=0$
3) $(x-1)(x+4)=0$
4) $0=(2x+5)(x-7)$
5) $6x^2-150=0$
6) $p^2+4p-32=0$
7) $2n^2+10n-28=0$
8) $m^2-m-30=0$
9) $x^2-4x-8=-8$
10) $v^2-8v-3=-3$
11) $x^2-5x-1=-5$
12) $a^2-6a+6=-2$
13) $7r^2+84=-49r$
14) $7m^2-224=28m$
15) $x^2-6x=16$
16) $7n^2-28n=0$
17) $3v^2=5v$
18) $2b^2=-3b$
19) $9x^2=30x-25$
20) $3n^2+39n=-36$
21) $4k^2+18k-23=6k-7$
22) $a^2+7a-9=-3+6a$
23) $9x^2-46+7x=7x+8x^2+3$
24) $x^2+10x+30=6$
25) $2m^2+19m+40=-5m$
26) $5n^2+45n+38=-2$
27) $5x(3x-6)-5x^2=x^2+6x+45$
28) $8x^2+11x-48=3x$
29) $41p^2+183p-196=183p+5p^2$
30) $121w^2+8w-7=8w-6$

4.6 Answers

1) $7, -2$
2) $-4, 3$
3) $1, -4$
4) $-\frac{5}{2}, 7$
5) $-5, 5$
6) $4, -8$
7) $2, -7$
8) $-5, 6$
9) $0, 4$
10) 0.8
11) $1, 4$
12) $2, 4$
13) $-4, -3$
14) $-4, 8$
15) $-2, 8$
16) $0, 4$
17) $0, \frac{5}{3}$
18) $-\frac{3}{2}, 0$
19) $\frac{5}{3}$
20) $-12, -1$
21) $-4, 1$
22) $2, -3$
23) $-7, 7$
24) $-4, -6$
25) $-10, -2$
26) $-8, -1$
27) $-1, 5$
28) $-3, 2$
29) $-\frac{7}{3}, \frac{7}{3}$
30) $-\frac{1}{11}, \frac{1}{11}$