Introductory Algebra with Statistics

MATH 082 Textbook

Third Edition (2020)

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Chapter 1: Review of Equations and Lines

Section 1.1: Review of Equations

Solve the equation for the variable.

When solving an equation for a variable, we do so by isolating the variable. That is, we move everything that is on the same side of the equal sign as the variable to the other side of the equation so that the variable is by itself. The following examples illustrate the various types of situations that might occur and what steps are needed to solve for the variable in each case.

Example 1.

$x+7=-5$ The 7 is added to the x

$\underline{\;-7\quad -7\;}$ Subtract 7 from both sides so that only x is on the left side

$x+0=-12$

$x=-12$ Our solution!

Example 2.

$x-5=4$ The 5 is negative, or subtracted from x

$\underline{\;+5\quad +5\;}$ Add 5 to both sides so that only x is on the left side

$x+0=9$

$x=9$ Our solution!

Example 3.

$-5x=30$ Variable is multiplied by -5

$\frac{-5x}{-5}=\frac{30}{-5}$ Divide both sides by -5, so that only x is on the left side

$1x=-6$

$x=-6$ Our Solution!

Example 4.

$\frac{x}{5}=-3$ Variable is divided by 5

$(5)\frac{x}{5}=-3(5)$ Multiply both sides by 5, so that only x is on the left side

$1x=-15$

$x=-15$ Our Solution!

Example 5.

$4-2x=10$	Start by focusing on the positive 4
$\underline{-4 \qquad -4}$	Subtract 4 from both sides
$\dfrac{-2x}{-2}=\dfrac{6}{-2}$	Negative (subtraction) stays on the $2x$ Divide by -2, the coefficient of $-2x$
$x=-3$	Our Solution!

Example 6.

$4(2x-6)=16$	Distribute 4 through parentheses
$8x-24=16$	Focus on the subtraction first
$\underline{\quad +24 \quad +24}$	Add 24 to both sides
$\dfrac{8x}{8}=\dfrac{40}{8}$	Notice the variable is multiplied by 8 Divide both sides by 8, the coefficient of $8x$
$x=5$	Our Solution!

Example 7.

$$4x-6=2x+10$$

Notice here the x is on both the left and right sides of the equation. This can make it difficult to decide which side to work with. We resolve this by moving one of the terms with x to the other side of the equation, much like we moved a constant term. It doesn't matter which term gets moved, $4x$ or $2x$.

$4x-6=2x+10$	Notice the variable on both sides
$\underline{-2x \qquad -2x}$	Subtract $2x$ from both sides
$2x-6=10$	Focus on the subtraction first
$\underline{\quad +6 \quad +6}$	Add 6 to both sides
$\dfrac{2x}{2}=\dfrac{16}{2}$	Notice the variable is multiplied by 2 Divide both sides by 2, the coefficient of $2x$
$x=8$	Our Solution!

Example 8.

$4(2x-6)+9=3(x-7)+8x$	Distribute 4 and 3 through parentheses
$8x-24+9=3x-21+8x$	Combine like terms $-24+9$ and $3x+8x$
$8x-15=11x-21$	Notice the variable is on both sides
$\underline{-8x \qquad -8x}$	Subtract $8x$ from both sides
$-15=3x-21$	Focus on subtraction of 21
$\underline{+21 \qquad +21}$	Add 21 to both sides
$6=3x$	Notice the variable is multiplied by 3
$\frac{6}{3}=\frac{3x}{3}$	Divide both sides by 3, the coefficient of $3x$
$2=x$	Our Solution!

Example 9.

$\frac{3}{4}x-\frac{7}{2}=\frac{5}{6}$	Focus on subtraction
$+\frac{7}{2} \quad +\frac{7}{2}$	Add $\frac{7}{2}$ to both sides

We will need to get a common denominator to add $\frac{5}{6}+\frac{7}{2}$. We have a common denominator of 6. So, we rewrite the fraction $\frac{7}{2}$ in terms of the common denominator by multiplying both the numerator and the denominator by 3, $\frac{7}{2}\left(\frac{3}{3}\right)=\frac{21}{6}$. We can now add the fractions:

$\frac{3}{4}x-\frac{21}{6}=\frac{5}{6}$	Same problem, with common denominator 6
$\underline{+\frac{21}{6} \quad +\frac{21}{6}}$	Add $\frac{21}{6}$ to both sides
$\frac{3}{4}x=\frac{26}{6}$	Reduce $\frac{26}{6}$ to $\frac{13}{3}$
$\frac{3}{4}x=\frac{13}{3}$	Focus on multiplication by $\frac{3}{4}$

We can get rid of $\frac{3}{4}$ by dividing both sides by $\frac{3}{4}$. Dividing by a fraction is the same as multiplying by the reciprocal, so we will multiply both sides by $\frac{4}{3}$.

$$\left(\frac{4}{3}\right)\frac{3}{4}x=\frac{13}{3}\left(\frac{4}{3}\right) \quad \text{Multiply by reciprocal}$$

$$x=\frac{52}{9} \quad \text{Our solution!}$$

While this process does help us arrive at the correct solution, the fractions can make the process quite difficult. This is why we use an alternate method for dealing with fractions – clearing fractions. We can easily clear the fractions by finding the LCD and multiplying each term by the LCD. This is shown in the next example, which is the same problem as our first example; but, this time we will solve by clearing the fractions.

Example 10.

$\frac{3}{4}x-\frac{7}{2}=\frac{5}{6}$	$\text{LCD}=12$, multiply each term by 12
$\frac{(12)3}{4}x-\frac{(12)7}{2}=\frac{(12)5}{6}$	Reduce the fractions
$(3)3x-(6)7=(2)5$	Multiply out each term
$9x-42=10$	Focus on subtraction by 42
$+42 \quad +42$	Add 42 to both sides
$9x=52$	Notice the variable is multiplied by 9
$\frac{9x}{9}=\frac{52}{9}$	Divide both sides by 9, the coefficient of $9x$
$x=\frac{52}{9}$	Our Solution!

World View Note: The study of algebra originally was called the "Cossic Art" from the Latin, the study of "things" (which we now call variables).

1.1 Practice

Solve each equation.

1) $v+9=16$
2) $14=b+3$
3) $x-11=-16$
4) $-14=x-18$
5) $340=-17x$
6) $4r=-28$
7) $-9=\frac{n}{12}$
8) $\frac{k}{13}=-16$
9) $24=2n-8$
10) $-5m+2=27$
11) $\frac{b}{3}+7=10$
12) $4+\frac{a}{3}=1$
13) $-21x+12=-6-3x$
14) $-1-7m=-8m+7$
15) $-7(x-2)=-4-6(x-1)$
16) $-6(x-8)-4(x-2)=-4$
17) $-2(8n-4)=8(1-n)$
18) $-4(1+a)=2a-8(5+3a)$
19) $\frac{3}{2}n-\frac{8}{3}=-\frac{29}{12}$
20) $\frac{3}{2}-\frac{7}{4}v=-\frac{9}{8}$
21) $\frac{45}{16}+\frac{3}{2}n=\frac{7}{4}n-\frac{19}{16}$
22) $\frac{2}{3}m+\frac{9}{4}=\frac{10}{3}-\frac{53}{18}m$

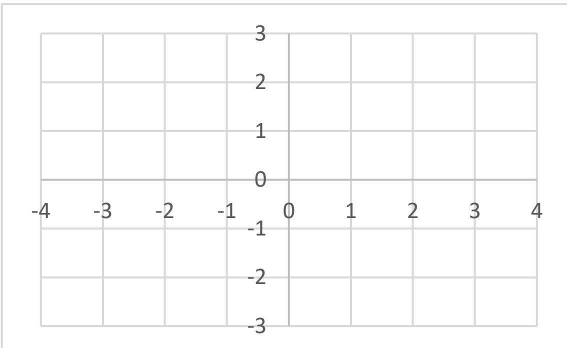
1.1 Answers

1) 7
2) 11
3) -5
4) 4
5) -20
6) -7
7) -108
8) -208
9) 16
10) -5
11) 9
12) -9
13) 1
14) 8
15) 12
16) 6
17) 0
18) -2
19) $\frac{1}{6}$
20) $\frac{3}{2}$
21) 16
22) $\frac{39}{130} = \frac{3}{10}$

Section 1.2: Points and Lines

Objective: Graph points and lines using x and y coordinates.

Often, to get an idea of the behavior of an equation we will make a picture that represents the solutions to the equations. A **graph** is simply a picture of the solutions to an equation. Before we spend much time on making a visual representation of an equation, we first have to understand the basics of graphing. The following is an example of what is called the coordinate plane.



The plane is divided into four sections by a horizontal number line (x -axis) and a vertical number line (y -axis). Where the two lines meet in the center is called the origin. This center origin is where $x=0$ and $y=0$. As we move to the right the numbers count up from zero, representing $x=1, 2, 3, \ldots$

To the left the numbers count down from zero, representing $x=-1, -2, -3, \ldots$. Similarly, as we move up the number count up from zero, $y=1, 2, 3, \ldots$, and as we move down count down from zero, $y=-1, -2, -3, \ldots$. We can put dots on the graph, which we will call points. Each point has an "address" that defines its location.

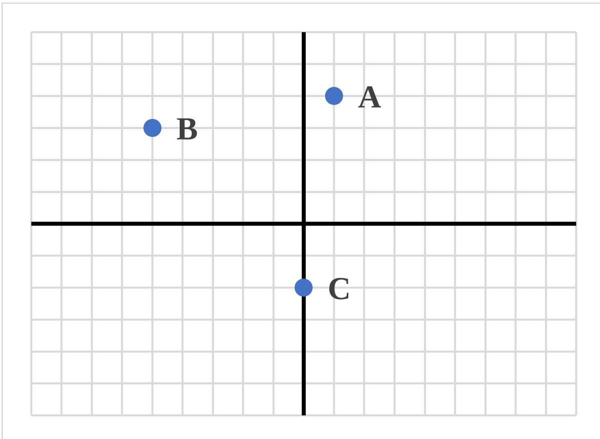
The first number will be the value on the x axis or horizontal number line. This is the distance the point moves left/right from the origin. The second number will represent the value on the y axis or vertical number line. This is the distance the point moves up/down from the origin. The points are given as an ordered pair: (x, y).

World View Note: Locations on the globe are given in the same manner. The longitude gives the distance east or west from a central point and is like the x value. The latitude gives the distance north or south of that central point and is like the y value. The central point is just off of the western coast of Africa where the equator and prime meridian meet.

The following example finds the address, or coordinate pair, for each of several points on the coordinate plane.

Example 1.

Give the coordinates of each point.



Tracing from the origin, point A is right 1, up 4. This becomes $A(1,4)$.

Point B is left 5, up 3. Left is backwards or negative, so we have $B(-5,3)$.

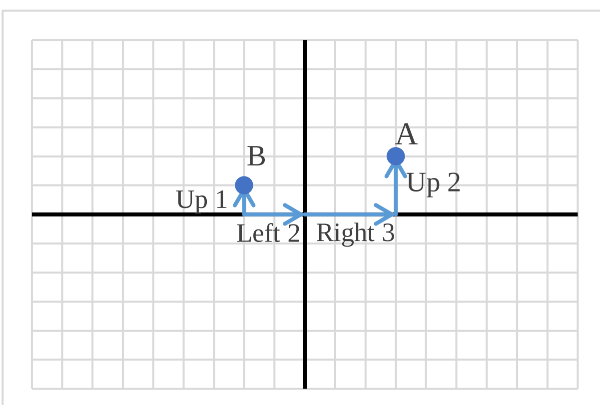
C is straight down 2 units. There is no left or right. This means we go right zero so the point is $C(0,-2)$

$A(1,4)$, $B(-5,3)$, $C(0,-2)$ Our Solution

Just as we can state the coordinates for a set of points, we can take a set of points and plot them on the plane.

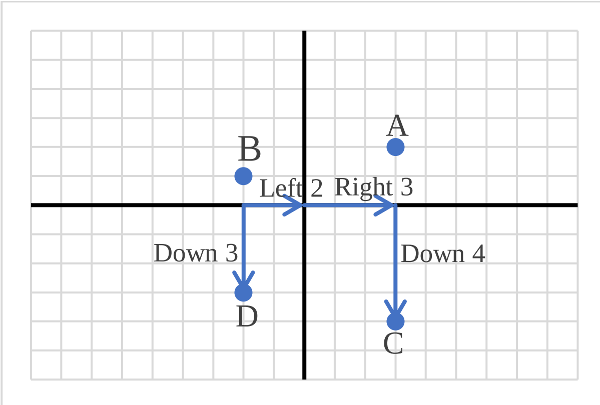
Example 2.

Graph the points $A(3,2)$, $B(-2,1)$, $C(3,-4)$, $D(-2,-3)$, $E(-3,0)$, $F(0,2)$, $G(0,0)$.



The first point, A is at $(3,2)$ this means $x=3$ (right 3) and $y=2$ (up 2). Following these instructions, starting from the origin, we get our point.

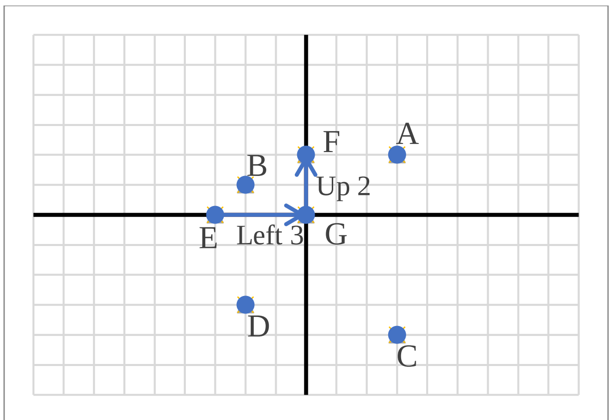
The second point, $B(-2,1)$, is left 2 (negative moves backwards), up 1. This is also illustrated on the graph.



The third point, $C(3,-4)$ is right 3, down 4 (negative moves backwards).

The fourth point, $D(-2,-3)$ is left 2, down 3 (both negative, both move backwards)

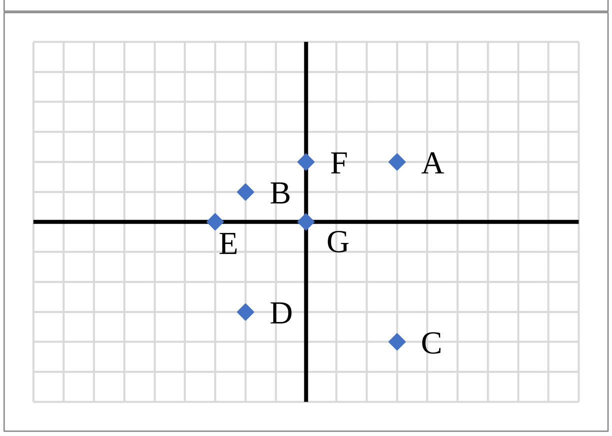
The last three points have zeros in them. We still treat these points just like the other points. If a coordinate is zero, then there is just no movement.



Next, graph $E(-3,0)$. This is left 3 (negative is backwards), and up zero, which is on the x-axis.

Then, graph $F(0,2)$. This is right zero, and up two, which is on the y-axis.

Finally, graph $G(0,0)$. This point has no movement. Thus, the point is exactly where the two axes meet, which is known as the origin.



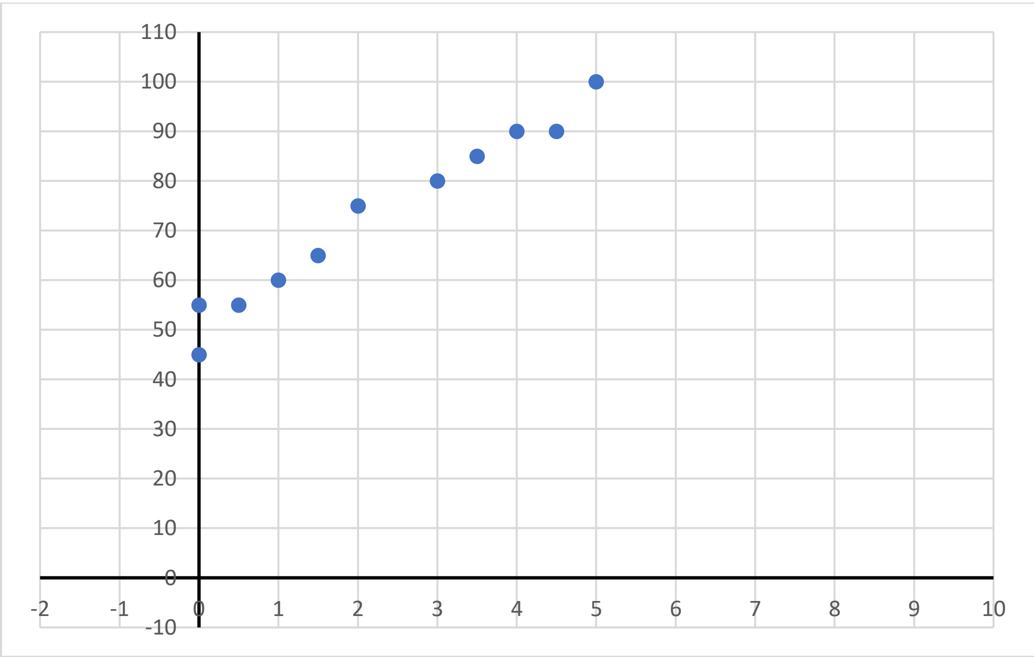
Our Solution

Example 3.

An instructor distributes the results of a midterm examination and then surveys the students to determine how many hours each student spent preparing for the exam. The results are summarized in the following table.

hours of preparation	exam score
0	45
0	55
0.5	55
1	60
1.5	65
2	75
3	80
3.5	85
4	90
4.5	90
5	100

We are given two variables: hours and scores. To visualize the impact the hours of preparation has on respective test scores, we can construct a graph. Each student can be represented by a point in the form (hours, score).



The graph illustrates the relationship between preparation hours and exam scores.

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. Consider $y = 2x - 3$. We may be interested in all of the possible solutions to this equation which involves a combination of an x and y value that make the equation true. Graphing can help visualize these solutions. We will do this using a table of values.

Example 4.

Graph $y = 2x - 3$. We make a table of values in which the first column is for the x values and the second column is for the y values.

x	y
-1	
0	
1	

We will test three values for x. Any three can be used

x	y
-1	-5
0	-3
1	-1

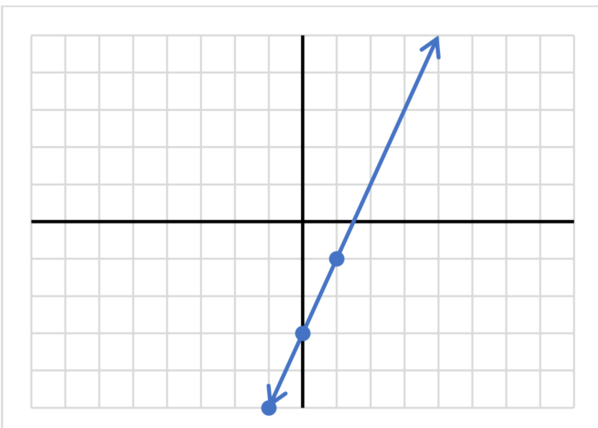
Evaluate each by replacing x with the given value

$x = -1$; $y = 2(-1) - 3 = -2 - 3 = -5$

$x = 0$; $y = 2(0) - 3 = 0 - 3 = -3$

$x = 1$; $y = 2(1) - 3 = 2 - 3 = -1$

$(-1, -5), (0, -3), (1, -1)$ These then become the points to graph on our equation



Plot each point.

Once the point are on the graph, connect the dots to make a line.

The graph is our solution

What this line tells us is that any point on the line will work in the equation $y = 2x - 3$. For example, notice the graph also goes through the point $(2, 1)$.

If we use $x = 2$, we should get $y = 1$. Sure enough, $y = 2(2) - 3 = 4 - 3 = 1$, just as the graph suggests. Thus, we have the line is a picture of all the solutions for $y = 2x - 3$. We can use this table of values method to draw a graph of any linear equation.

Example 5.

Graph $2x - 3y = 6$. We will use a table of values.

x	y
-3	
0	
3	

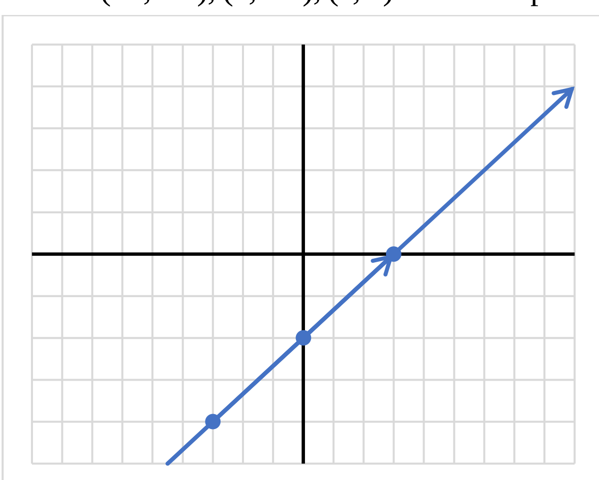
We will test three values for x. Any three can be used

$2(-3)-3y=6$	Substitute each value in for x and solve for y
$-6-3y=6$	Start with $x=-3$, multiply first
$+6 \quad +6$	Add 6 to both sides
$-3y=12$	
$\dfrac{-3y}{-3}=\dfrac{12}{-3}$	Divide both sides by -3
$y=-4$	Solution for y when $x=-3$, add this to table
$2(0)-3y=6$	Next $x=0$
$-3y=6$	Multiplying clears the constant term
$\dfrac{-3y}{-3}=\dfrac{6}{-3}$	Divide each side by -3
$y=-2$	Solution for y when $x=0$, add this to table
$2(3)-3y=6$	Next $x=3$
$6-3y=6$	Multiply
$-6 \quad -6$	Subtract -6 from both sides
$-3y=0$	
$\dfrac{-3y}{-3}=\dfrac{0}{-3}$	Divide each side by -3
$y=0$	Solution for y when $x=-3$, add this to table

x	y
-3	-4
0	-2
3	0

Out completed table

$(-3,-4), (0,-2), (3,0)$ Our completed table represents ordered pairs to plot.



Graph points and connect the dots

Our Solution

Example 6.

Graph $y=\frac{2}{3}x-1$. We will use a table of values.

x	y
0	
3	
6	

We will test three values for x. Note that in this case, choosing multiples of 3 will eliminate the fraction.

$y=\frac{2}{3}(0)-1$ Substitute each value in for x and solve for y

$y=0-1$ Multiply first, now subtract.

$y=-1$ Solution for y when $x=0$, add this to table

$y=\frac{2}{3}(3)-1$ Next, $x=3$.

$y=\frac{6}{3}-1$ Multiply first, now divide

$y=2-1$ Subtract

$y=1$ Solution for y when $x=3$, add this to table

$y=\frac{2}{3}(6)-1$ Next, $x=6$

$y=\frac{12}{3}-1$ Multiply first, now divide

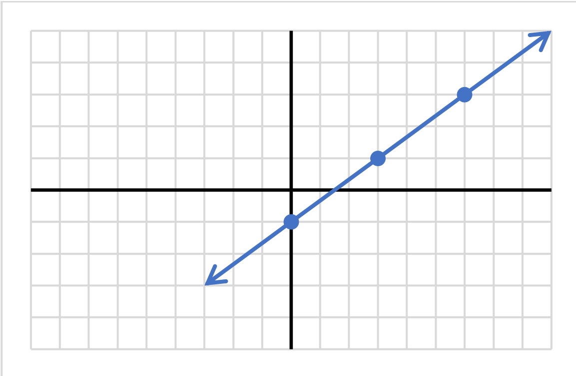
$y=4-1$ Subtract

$y=3$ Solution for y when $x=6$, add this to table

x	y
0	-1
3	1
6	3

Our completed table.

$(0,-1),(3,1),(6,3)$ Our table represents ordered pairs to plot:



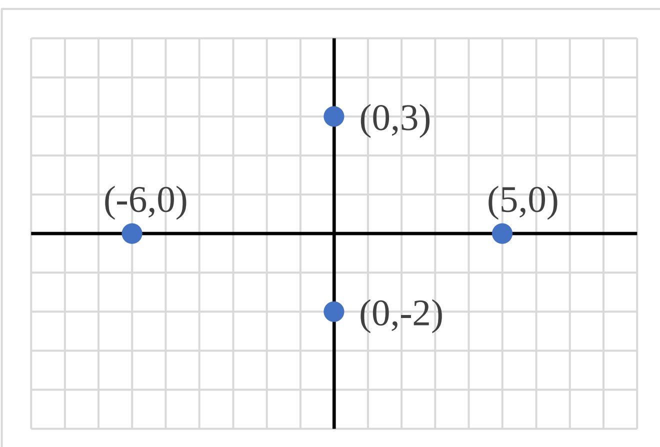
Graph points and connect dots

Our Solution

Objective: Graph lines using intercepts.

In the previous examples we constructed graphs of lines by plotting randomly selected points. Now we will find specific points on the graph of a line: $x-$and $y-$intercepts. The **x – intercept** of the graph of a line is the point where the line intersects the x axis (horizontal axis). The **y – intercept** of the graph of a line is the point where the line intersects the y axis (vertical axis).

Before we determine the location of these intercepts, given a particular equation, examine the following graph. What do you notice?



Points along the $x-$axis have $y=0$, and points along the $y-$axis have $x=0$.

Therefore, to determine the intercepts, we will substitute values of 0 into the given equation one variable at a time.

Example 7.

Determine the $x-$and $y-$intercepts and sketch the graph of $9x-6y=18$.

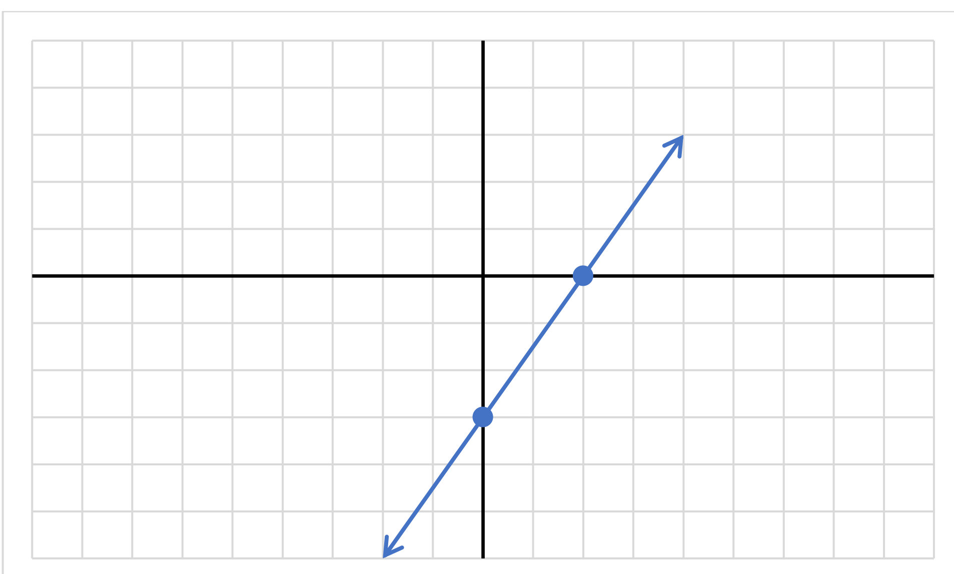
First, we determine the $x-$intercept.

$9x-6y=18$	Our original equation.
$9x-6(0)=18$	Let $y=0$.
$9x=18$	
$\frac{9x}{9}=\frac{18}{9}$	Divide each side by 9.
$x=2$	
$(2, 0)$	Write as an ordered pair.

Second, we determine the $y-$intercept.

$9x-6y=18$	Our original equation.
$9(0)-6y=18$	Let $x=0$.
$-6y=18$	
$\frac{-6y}{-6}=\frac{18}{-6}$	Divide each side by -6.
$y=-3$	
$(0,-3)$	Write as an ordered pair.

Now we can sketch the graph of the line by plotting the intercepts.



Example 8.

Determine the $x-$and $y-$intercepts and sketch the graph of $y=\frac{3}{5}x-2$.

First, we determine the $x-$intercept.

$y=\frac{3}{5}x-2$ Our original equation.

Let $y=0$.

$0=\frac{3}{5}x-2$

Solve for x. Add 2 to both sides.

$0=\frac{3}{5}x-2$

$$\begin{array}{r} 0=\frac{3}{5}x-2 \\ \underline{+2 \qquad +2} \\ 2=\frac{3}{5}x \end{array}$$

Multiply both sides by $\frac{5}{3}$.

$\frac{5}{3}\cdot 2=\frac{5}{3}\cdot\frac{3}{5}x$

$\frac{10}{3}=x$

$\left(\frac{10}{3},0\right)$ or $\left(3\frac{1}{3},0\right)$ Write as an ordered pair.

Second, we determine the $y-$intercept.

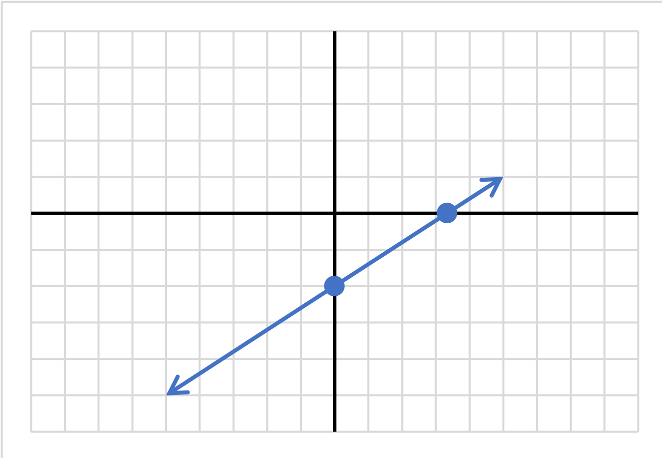
$y=\frac{3}{5}x-2$ Our original equation.

$y=\frac{5}{3}(0)-2$ Let $x=0$

$y=-2$

$(0,-2)$ Write as an ordered pair

Now we can sketch the graph of the line by plotting the intercepts.

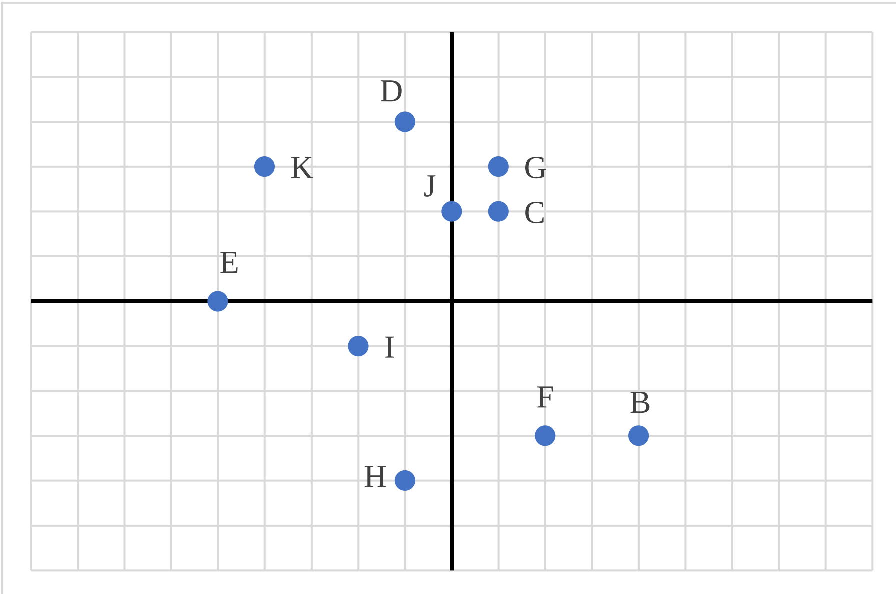


Improper fractions are sometimes better expressed as mixed numbers so they can be estimated on the graph.

The $x-$intercept of $\left(\frac{10}{3}, 0\right)$ is expressed as $\left(3\frac{1}{3}, 0\right)$ and its point is estimated, slightly to the right of $+3$ along the $x-$axis.

1.2 Practice

State the coordinates of each point in the graph.

1) 

Plot each point on a graph.

2) $C(0,4)$ $K(1,0)$ $J(-3,4)$ $I(-3,0)$ $H(-4,2)$ $G(4,-2)$ $F(-2,-2)$ $E(3,-2)$ $D(0,3)$ $L(-5,5)$

Sketch the graph of each line by plotting points.

3) $y=-2x-3$
4) $y=5x-4$
5) $y=-4x+2$
6) $y=\frac{3}{2}x-5$
7) $y=\frac{4}{5}x-3$
8) $y=-x-5$
9) $4x+y=5$
10) $2x-y=2$
11) $x+y=-1$
12) $x-y=-3$
13) $y=3x+1$

14) $y=\frac{5}{3}x+4$

15) $y=-x-2$

16) $y=\frac{1}{2}x$

17) $8x-y=5$

Determine the $\mathbf{x}-$ and $\mathbf{y}-$ intercepts, and sketch the graph of each line.

18) $2x+5y=10$

19) $2x+6y=18$

20) $x+3y=6$

21) $4x-y=4$

22) $x-7y=7$

23) $x+y=6$

24) $x+y=3$

25) $y=5x-10$

26) $y=4x+12$

27) $y=x+8$

28) $y=-x-6$

29) $y=\frac{1}{2}x+3$

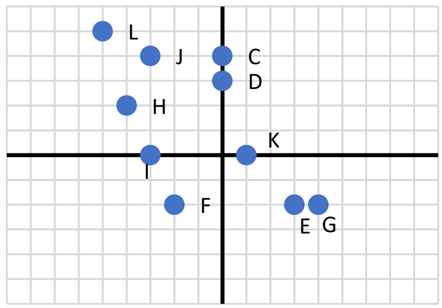
30) $y=\frac{3}{4}x-2$

31) $3x+4y=16$

1.2 Answers

1) $B(4,-3)$ $C(1,2)$ $D(-1,4)$ $E(-5,0)$ $F(2,-3)$ $G(1,3)$ $H(-1,-4)$ $I(-2,-1)$ $J(0,2)$ $K(-4,3)$

2)



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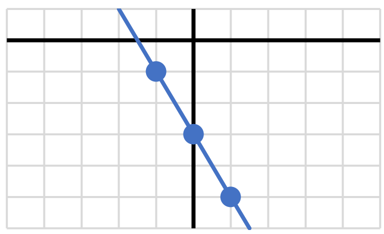

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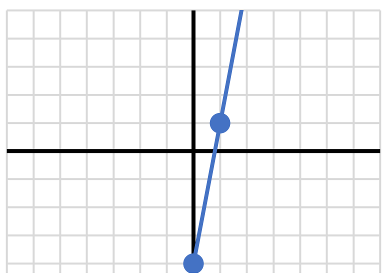

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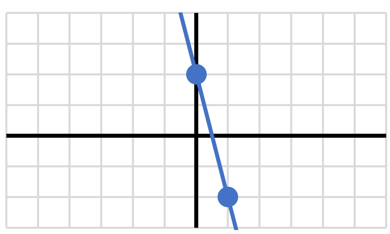

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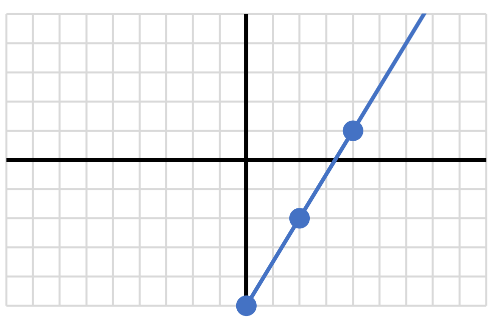

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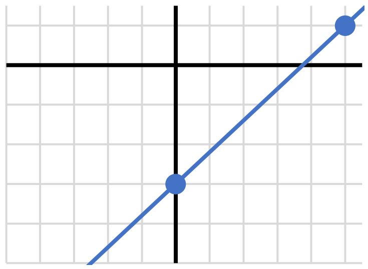

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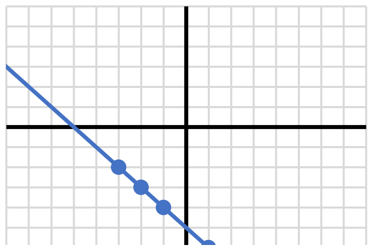

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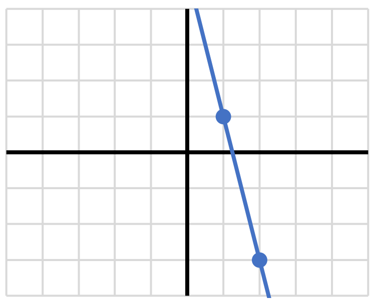

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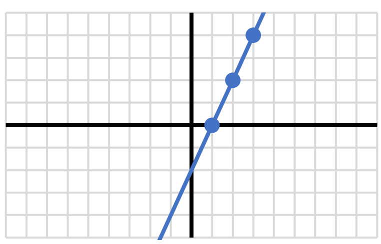

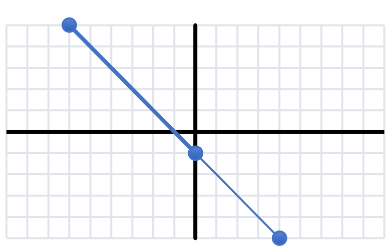

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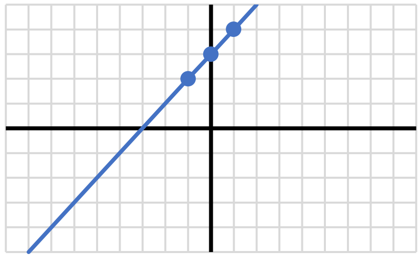

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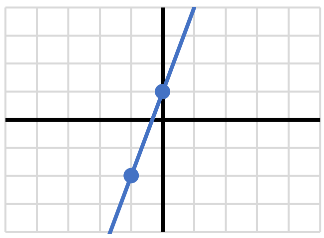

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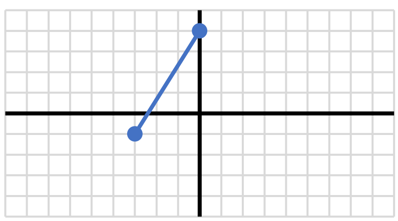

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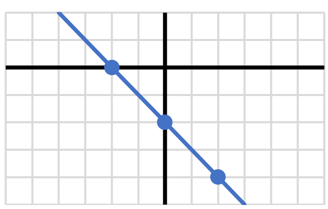

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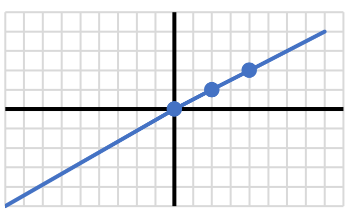

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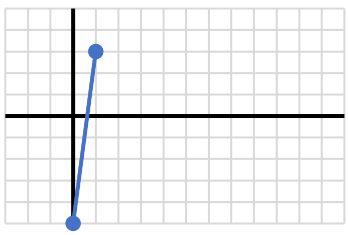

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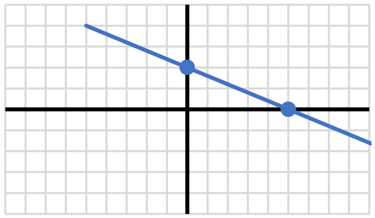

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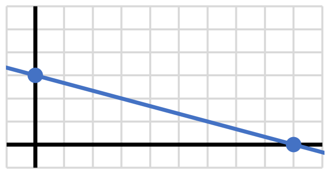

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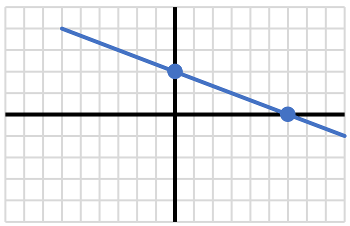

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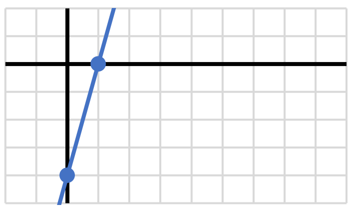

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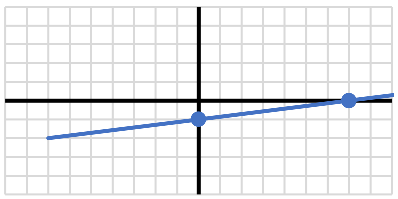

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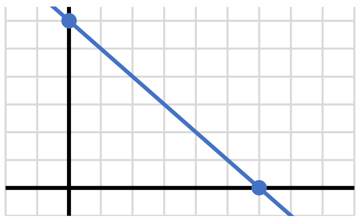

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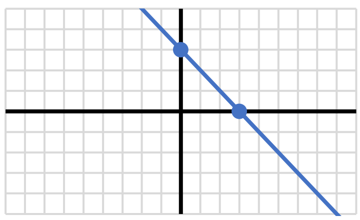
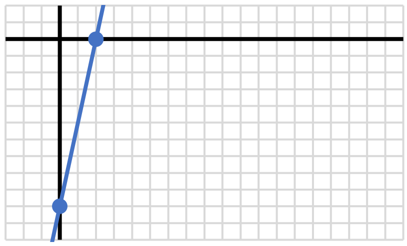
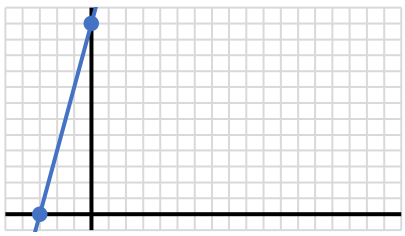
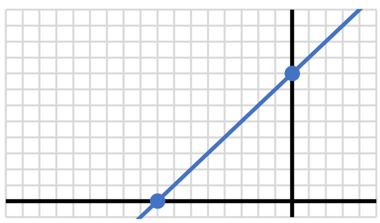
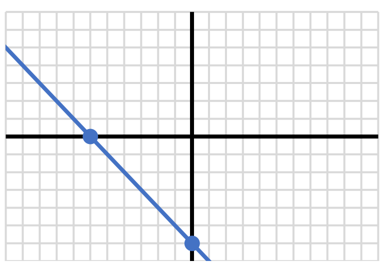
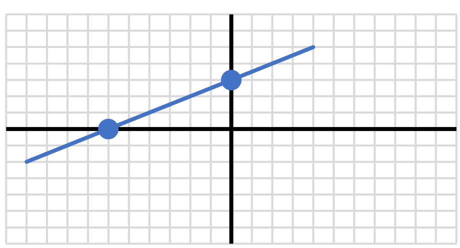


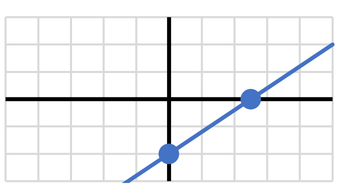
25) 

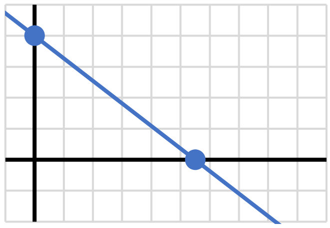
26) 

27) 

28) 

29) 

30) 

31) 

Section 1.3: Slope

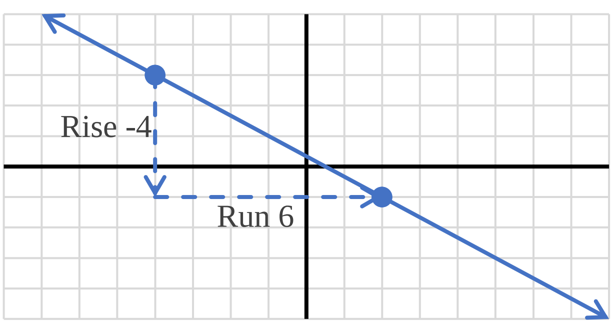
Objective: Find the slope of a line given a graph or two points.

As we graph lines, we will want to be able to identify different properties of the lines we graph. One of the most important properties of a line is its slope. **Slope** is a measure of steepness. A line with a large slope, such as 25, is very steep. A line with a small slope, such as $\frac{1}{10}$ is very flat. We will also use slope to describe the direction of the line. A line that goes up from left to right will have a positive slope and a line that goes down from left to right will have a negative slope.

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction $\frac{\text{rise}}{\text{run}}$.

Rise would be a vertical change, or a change in the $y-$values. Run would be a horizontal change, or a change in the $x-$values. So another way to describe slope would be the fraction $\frac{\text{change in } y}{\text{change in } x}$. It turns out that if we have a graph we can draw vertical and horizontal lines from one point to another to make what is called a slope triangle. The sides of the slope triangle give us our slope. The following examples show graphs that we find the slope of using this idea.

Example 1.



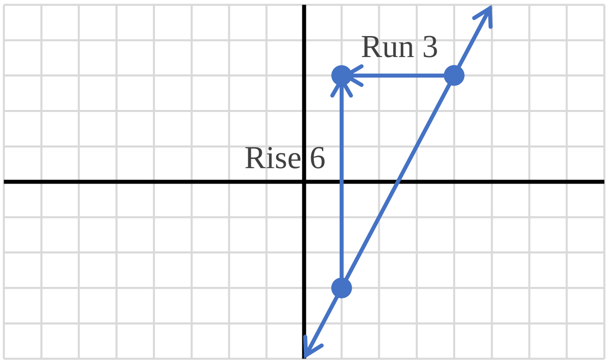
To find the slope of this line we will consider the rise, or vertical change and the run or horizontal change. Drawing these lines in makes a slope triangle that we can use to count from one point to the next the graph goes down 4, right 6. This is rise -4, run 6. As a fraction it would be, $\frac{-4}{6}$. Reduce the fraction to get $\frac{-2}{3}$.

$$\text{Slope} = -\frac{2}{3}$$

Our Solution

World View Note: When French mathematicians Rene Descartes and Pierre de Fermat first developed the coordinate plane and the idea of graphing lines (and other functions) the $y-$axis was not a vertical line!

Example 2.



To find the slope of this line, the rise is up 6, the run is right 3. Our slope is then written as a fraction, $\frac{\text{rise}}{\text{run}}$ or

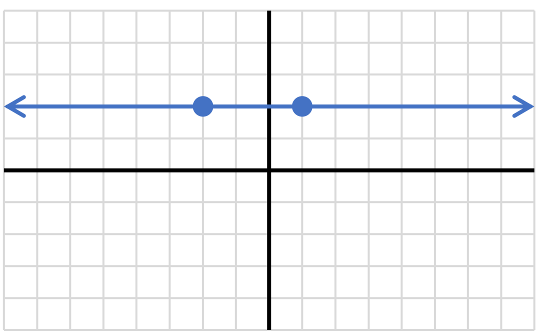
$$\frac{6}{3}$$

This fraction reduces to 2. This will be our slope.

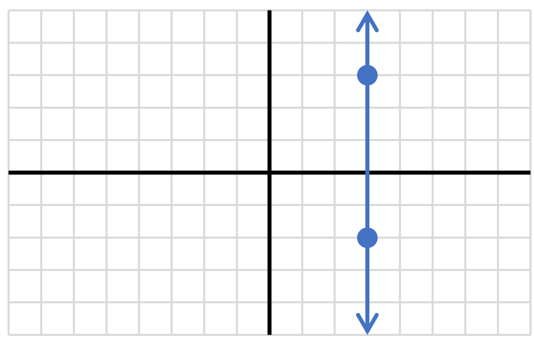
Slope $= 2$ Our Solution

There are two special lines that have unique slopes that we need to be aware of. They are illustrated in the following example.

Example 3.



This line has no rise, but the run is 3 units. The slope becomes $\frac{0}{3} = 0$.



This line has a rise of 5, but no run. The slope becomes $\frac{5}{0} =$ undefined.

We generalize the previous example and state that all horizontal lines have 0 slope and all vertical lines have undefined slope.

As you can see there is a big difference between having a zero slope and having an undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all, in fact it is flat. Therefore, it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express how steep it is. This is an undefined slope.

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y values, we can calculate this by subtracting the y values of a point. Similarly, if run is a change in the x values, we can calculate this by subtracting the x values of a point. In this way we get the following equation for slope.

$$\textbf{The slope of a line through } (x_1, y_1) \textbf{ and } (x_2, y_2) \textbf{ is } \frac{y_2 - y_1}{x_2 - x_1}$$

When mathematicians began working with slope, it was called the modular slope. For this reason we often represent the slope with the variable m. Now we have the following for slope.

$$\textbf{Slope} = m = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } y}{\textbf{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As we subtract the y values and the x values when calculating slope it is important we subtract them in the same order. This process is shown in the following examples.

Example 4.

Find the slope between $(-4, 3)$ and $(2, -9)$	Identify x_1, y_1, x_2, y_2
(x_1, y_1) and (x_2, y_2)	Use slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$
$m = \frac{-9-3}{2-(-4)}$	Simplify
$m = \frac{-12}{6}$	Reduce
$m = -2$	Our Solution

Example 5.

Find the slope between $(4, 6)$ and $(2, -1)$	Identify x_1, y_1, x_2, y_2
(x_1, y_1) and (x_2, y_2)	Use slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$
$m = \frac{-1-6}{2-4}$	Simplify
$m = \frac{-7}{-2}$	Reduce, dividing by -1
$m = \frac{7}{2}$	Our Solution

We may come up against a problem that has a zero slope (horizontal line) or no slope (vertical line) just as with using the graphs.

Example 6.

Find the slope between $(-4,-1)$ and $(-4,-5)$	Identify x_1, y_1, x_2, y_2
(x_1, y_1) and (x_2, y_2)	Use slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
$m = \dfrac{-5-(-1)}{-4-(-4)}$	Simplify
$m = \dfrac{-4}{0}$	Can’t divide by zero, undefined
$m = \text{undefined}$	Our Solution

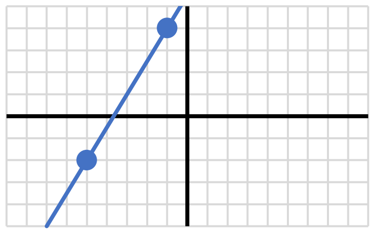
Example 7.

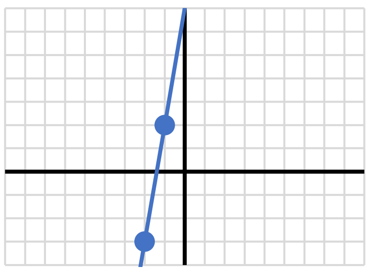
Find the slope between $(3,1)$ and $(-2,1)$	Identify x_1, y_1, x_2, y_2
(x_1, y_1) and (x_2, y_2)	Use slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
$m = \dfrac{1-1}{-2-3}$	Simplify
$m = \dfrac{0}{-5}$	Reduce
$m = 0$	Our Solution

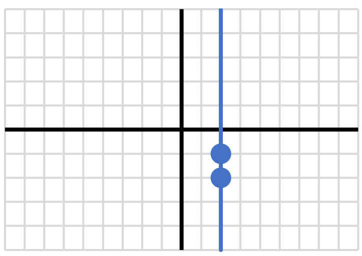
Again, there is a big difference between an undefined slope and a zero slope. Zero is an integer and it has a value, 0, is the slope of a horizontal line. Undefined slope has no value, it is undefined, and is the slope of a vertical line.

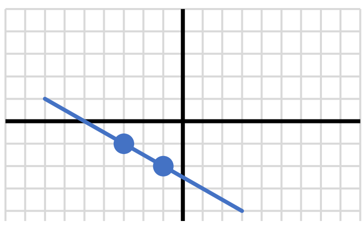
1.3 Practice

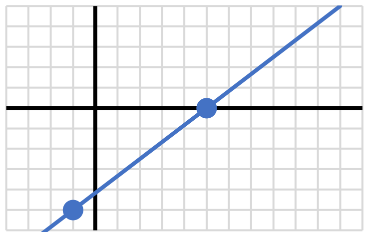
Find the slope of each line.

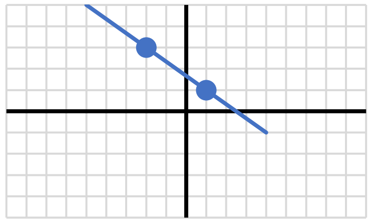
1) 

2) 

3) 

4) 

5) 

6) 

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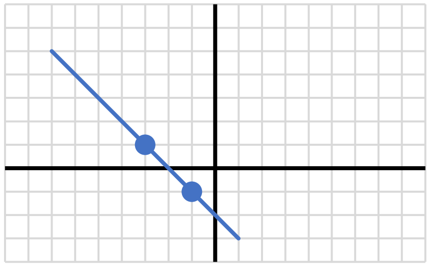

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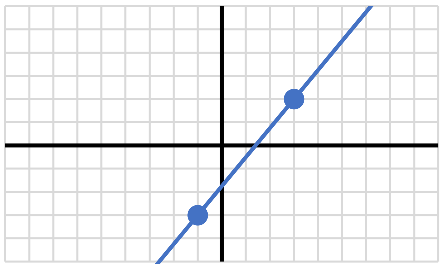

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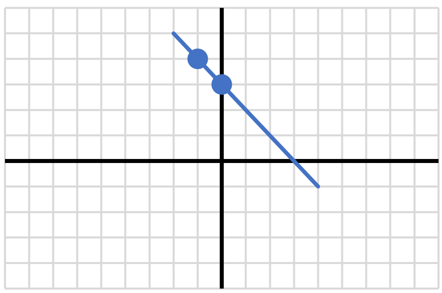

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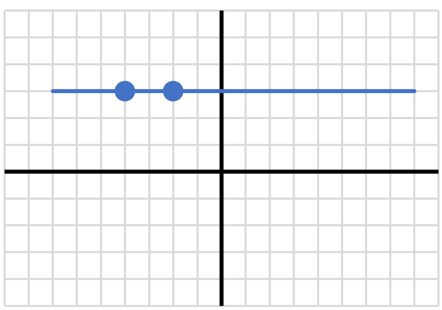


11) $(-2,10),(-2,-15)$
12) $(1,2),(-6,-14)$
13) $(-15,10),(16,-7)$
14) $(13,-2),(7,7)$
15) $(10,18),(-11,-10)$
16) $(-3,6),(-20,13)$
17) $(-16,-14),(11,-14)$
18) $(13,15),(2,10)$
19) $(-4,14),(-16,8)$
20) $(9,-6),(-7,-7)$
21) $(12,-19),(6,14)$

22) $(-5,-10), (-5,20)$
23) $(-16,2), (15,-10)$
24) $(8,11), (-3,-13)$
25) $(-17,19), (10,-7)$
26) $(11,-2), (1,17)$
27) $(7,-14), (-8,-9)$
28) $(-18,-5), (14,-3)$
29) $(-5,7), (-18,14)$
30) $(19,15), (5,11)$

1.3 Answers

1) $\frac{3}{2}$
2) 5
3) Undefined
4) $-\frac{1}{2}$
5) $\frac{5}{6}$
6) $-\frac{2}{3}$
7) -1
8) $\frac{5}{4}$
9) -1
10) 0
11) Undefined
12) $\frac{16}{7}$
13) $-\frac{17}{31}$
14) $-\frac{3}{2}$
15) $\frac{4}{3}$
16) $-\frac{7}{17}$
17) 0
18) $\frac{5}{11}$
19) $\frac{1}{2}$
20) $\frac{1}{16}$
21) $-\frac{11}{2}$
22) Undefined
23) $-\frac{12}{31}$
24) $\frac{24}{11}$
25) $-\frac{26}{27}$
26) $-\frac{19}{10}$
27) $-\frac{1}{3}$
28) $\frac{1}{16}$
29) $-\frac{7}{13}$
30) $\frac{2}{7}$

Section 1.4: Slope-Intercept Form

Objective: Give the equation of a line with a known slope and *y*-intercept.

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the $y-$intercept of the equation. The slope can be represented by m. The $y-$intercept, where the line crosses the $y-$axis can be represented by $(0, b)$ where b is the value where the graph crosses the vertical $y-$axis (thus, the x-coordinate is zero). Any other point on the line can be represented by (x, y). Using this information we will look at the slope formula and solve the formula for y.

Example 1.

$m, (0, b), (x, y)$	Using the slope formula gives:
$\dfrac{y-b}{x-0}=m$	Simplify
$\dfrac{y-b}{x}=m$	Multiply both sides by *x*
$y-b=mx$	Add *b* to both sides
$\underline{\quad +b \quad +b}$	
$y=mx+b$	Our Solution

This equation, $y=mx+b$ can be thought of as the equation of any line that has a slope of m and a $y-$intercept of b. This formula is known as the slope-intercept equation.

Slope-Intercept Equation: $\boldsymbol{y=mx+b}$

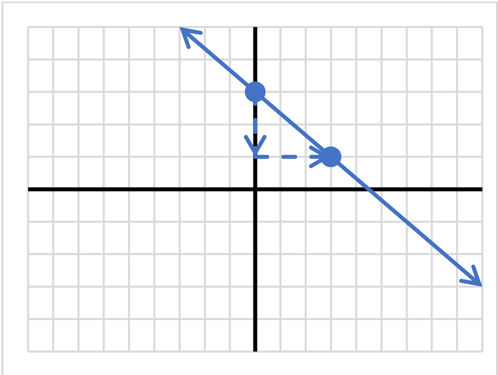
If we know the slope and the $y-$intercept we can easily find the equation that represents the line.

Example 2.

$\text{slope}=\dfrac{3}{4}, y-\text{intercept at } (0,-3)$	Use the slope – intercept equation
$y=mx+b$	*m* is the slope, *b* is the *y*-intercept
$y=\dfrac{3}{4}x-3$	Our Solution

We can also find the equation by looking at a graph and finding the slope and $y-$intercept.

Example 3.



Identify the point where the graph crosses the $y-$ axis $(0,3)$. This means the $y-$intercept is 3.
Draw a slope triangle to identify another point. The slope is $-\frac{2}{3}$

$y = mx + b$ Slope – intercept equation

$y = -\dfrac{2}{3}x + 3$ Our Solution

We can also move the opposite direction, using the equation to identify the slope and $y-$ intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for y so we can identify the slope and the y-intercept.

Example 4.

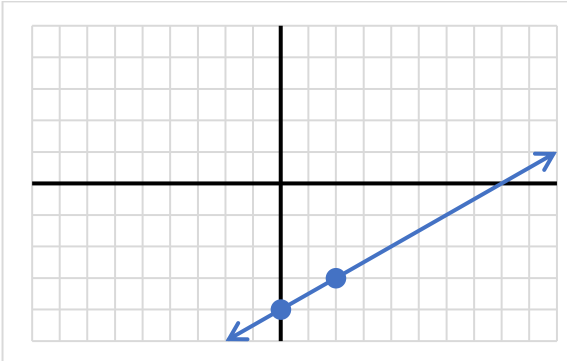
Write in slope – intercept form:

$2x - 4y = 6$	Solve for y
$-2x \quad\quad -2x$	Subtract $2x$ from both sides
	Put x term first
$\dfrac{-4y}{-4} = \dfrac{-2x}{-4} + \dfrac{6}{-4}$	Divide each term by –4
$y = \dfrac{1}{2}x - \dfrac{3}{2}$	Our Solution

Once we have an equation in slope-intercept form we can graph it by first plotting the $y-$ intercept, then using the slope, find a second point and connecting the dots.

Example 5.

Graph $y = \dfrac{1}{2}x - 4$	Recall the slope – intercept formula
$y = mx + b$	Identify the slope, m, and the y-intercept, b
$m = \dfrac{1}{2},\ b = -4$	Make the graph

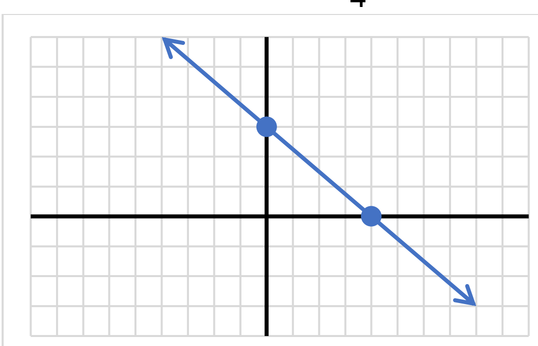


Start with a point at the $y-$ intercept of -4.
Then use the slope $\frac{\text{rise}}{\text{run}}$, so we will rise 1 unit and run 2 units to find the next point.
Once we have both points, connect the dots to get our graph.

World View Note: Before our current system of graphing, in 1323 French mathematician Nicole Oresme, suggested graphing lines that would look more like a bar graph with a constant slope!

Example 6.

Graph $3x+4y=12$	Not in slope intercept form
$-3x \quad\quad -3x$	Subtract $3x$ from both sides
$4y=-3x+12$	Put the x term first
$\frac{4y}{4}=\frac{-3x}{4}+\frac{12}{4}$	Divide each term by 4
$y=-\frac{3}{4}x+3$	Recall slope – intercept equation
$y=mx+b$	Identify m and b
$m=-\frac{3}{4}, b=3$	Make the graph



Start with a point at the $y-$ intercept of 3.
Then use the slope $\frac{\text{rise}}{\text{run}}$, but it's negative so it will go downhill, so we will drop 3 units and run 4 units to find the next point.
Once we have both points, connect the dots to get our graph.

We want to be very careful not to confuse using slope to find the next point with using a coordinate such as $(4,-2)$ to find an individual point. Coordinates such as $(4,-2)$ start from the origin and move horizontally first, and vertically second.

Slope starts from a point on the line that could be anywhere on the graph. The numerator is the vertical change and the denominator is the horizontal change.

Example 7.

A driving service charges an initial service fee of $6 and an additional $3 per mile traveled. Construct an equation that expresses the total cost, y, for traveling x miles. Identify the slope and $y-$intercept, and their meaning in context to this problem.

It may be helpful to calculate the total cost for several cases. The $6 service fee is constant; every traveler will pay at least $6. Added to this service fee is $3 for every mile. The total costs for three cases follow.

Mileage	Total cost
7	$6+3(7)=\$27$
8	$6+3(8)=\$30$
9	$6+3(9)=\$33$

If y represents the total charge and x represents the mileage, this can be generalized as

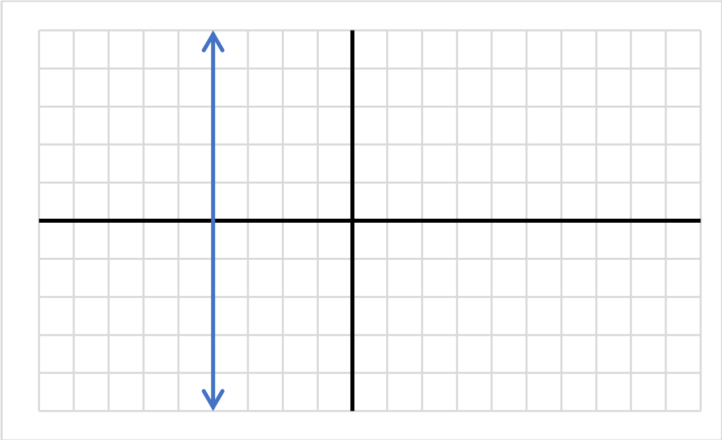
$$y=6+3x$$

Therefore, the slope is 3 and $y-$intercept is $(0,6)$. The slope represents the average rate of change. As the mileage increases by 1 mile, the total cost increases by $3: This can be seen in the examples given when traveling 7, 8, or 9 miles: The $y-$intercept represents the case where x, or mileage, is 0. When miles traveled is 0, meaning you just entered the vehicle, your cost is $6.

Recall that a horizontal line has slope equal to 0. Replacing *m* with 0 in the slope-intercept equation gives the equation $y=0x+b$ or just $y=b$. So, the equation of any horizontal line is of the form $y=b$, where *b* is the y-intercept of the line.

Recall that a vertical line has undefined slope; so, we cannot use the slope-intercept form of the equation at all. The equation of any vertical line is $x=a$, where *a* is the x-intercept of the line.

Example 8.



Give the equation of the line in the graph.

Because we have a vertical line, the slope is undefined; therefore, we cannot use the slope – intercept equation. Instead, we use the $x-$intercept of -4 to write the equation of the line.

$x=-4$ Our Solution

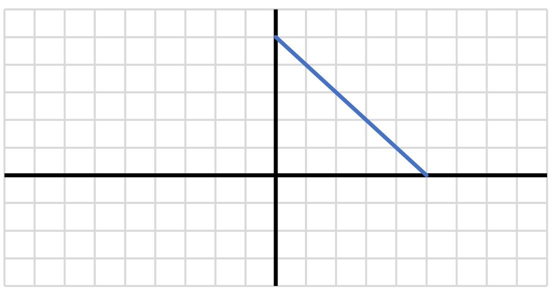
1.4 Practice

Write the slope-intercept form of the equation for each line when given the slope and the *y*-intercept.

1) $\text{Slope} = 2,\ y-\text{intercept} = 5$
2) $\text{Slope} = -6,\ y-\text{intercept} = 4$
3) $\text{Slope} = 1,\ y-\text{intercept} = -4$
4) $\text{Slope} = -1,\ y-\text{intercept} = -2$
5) $\text{Slope} = -\frac{3}{4},\ y-\text{intercept} = -1$
6) $\text{Slope} = -\frac{1}{4},\ y-\text{intercept} = 3$
7) $\text{Slope} = \frac{1}{3},\ y-\text{intercept} = 1$
8) $\text{Slope} = \frac{2}{5},\ y-\text{intercept} = 5$

Write the slope-intercept form of the equation of each line graphed below.

9)



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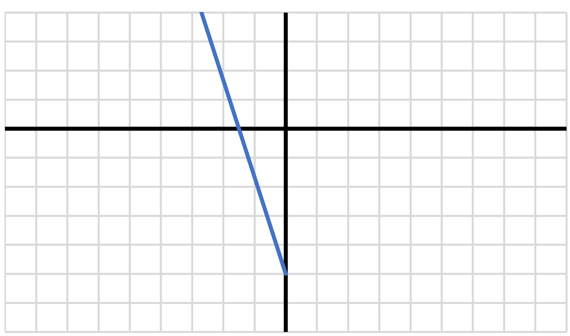

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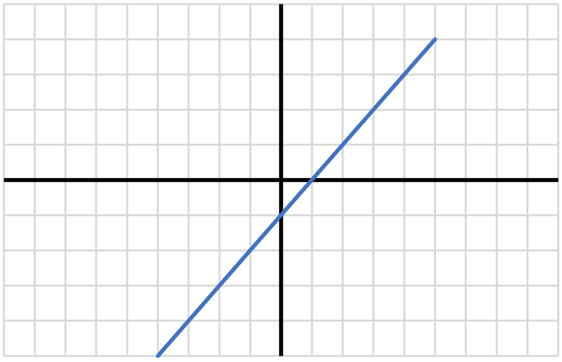

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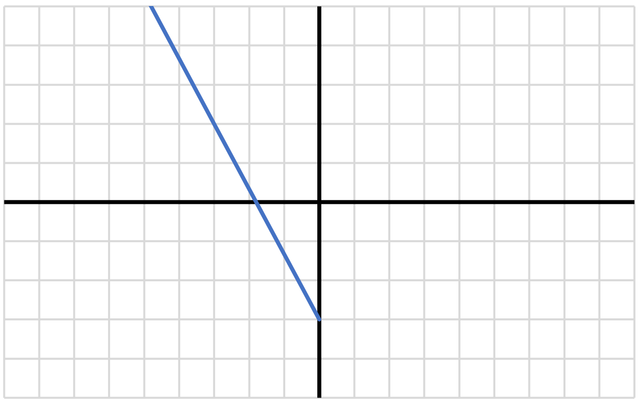

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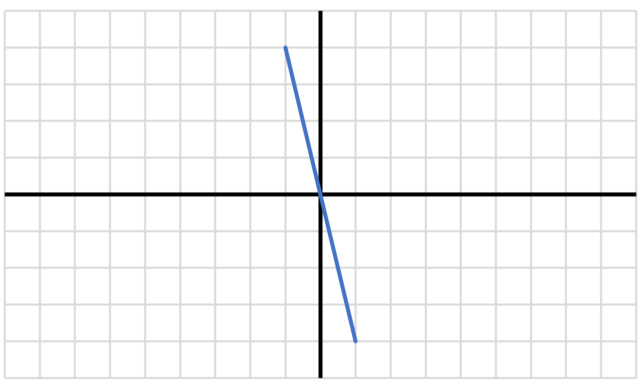

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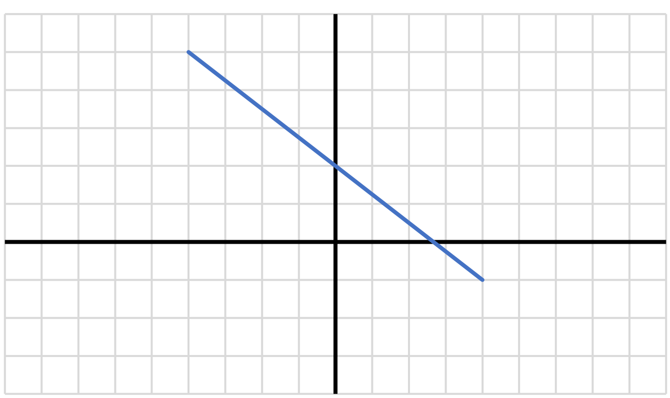


Write the equation of each line using the slope-intercept form.

15) $x+10y=-37$
16) $x-10y=3$
17) $2x+y=-1$
18) $6x-11y=-70$
19) $7x-3y=24$
20) $4x+7y=28$
21) $x-7y=-42$
22) $y-4=-(x+5)$
23) $y-5=\frac{5}{2}(x-2)$
24) $y-4=4(x-1)$
25) $y-3=-\frac{2}{3}(x+3)$
26) $y+5=-4(x-2)$
27) $y+1=-\frac{1}{2}(x-4)$
28) $y+2=\frac{6}{5}(x+5)$

Sketch the graph of each line.

29) $y=\frac{1}{3}x+4$
30) $y=-\frac{1}{5}x-4$
31) $y=\frac{6}{5}x-5$
32) $y=-\frac{3}{2}x-1$
33) $y=\frac{3}{2}x$
34) $y=-\frac{3}{4}x+1$
35) $x-y+3=0$
36) $4x+5=5y$
37) $-y-4+3x=0$
38) $-8=6x-2y$
39) $-3y=-5x+9$
40) $-3y=3-\frac{3}{2}x$

Consider each scenario and develop an applicable model.

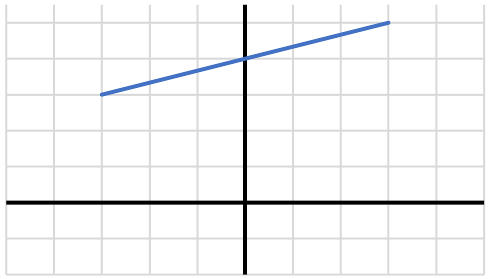
41) The initial room temperature of a beverage is 70°F. When placed in a particular refrigerator, the beverage is expected to cool (or decrease temperature) by an average of 5°F per hour. Express the temperature of the beverage, *y*, after remaining in the refrigerator for *x* hours. Identify the slope and *y*-intercept, and identify their meaning in context to this problem.

42) A reloadable banking card has an initial cost of \$4.95 and a service fee of \$2.95 per month. Express the total cost, *y*, of maintaining this banking card for *x* months. Identify the slope and *y*-intercept, and identify their meaning in context to this problem.

1.4 Answers

1) $y=2x+5$
2) $y=-6x+4$
3) $y=x-4$
4) $y=-x-2$
5) $y=-\frac{3}{4}x-1$
6) $y=-\frac{1}{4}x+3$
7) $y=\frac{1}{3}x+1$
8) $y=\frac{2}{5}x+5$
9) $y=-x+5$
10) $y=-\frac{7}{2}x-5$
11) $y=x-1$
12) $y=-\frac{5}{3}x-3$
13) $y=-4x$
14) $y=-\frac{3}{4}x+2$
15) $y=-\frac{1}{10}x-\frac{37}{10}$
16) $y=\frac{1}{10}x-\frac{3}{10}$
17) $y=-2x-1$
18) $y=\frac{6}{11}x+\frac{70}{11}$
19) $y=\frac{7}{3}x-8$
20) $y=-\frac{4}{7}x+4$
21) $y=\frac{1}{7}x+6$
22) $y=-x-1$
23) $y=\frac{5}{2}x$
24) $y=4x$
25) $y=-\frac{2}{3}x+1$
26) $y=-4x+3$
27) $y=-\frac{1}{2}x+1$
28) $y=\frac{6}{5}x+4$

29)



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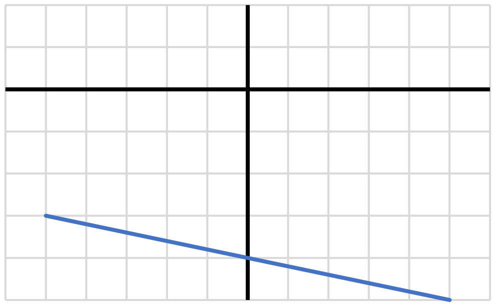

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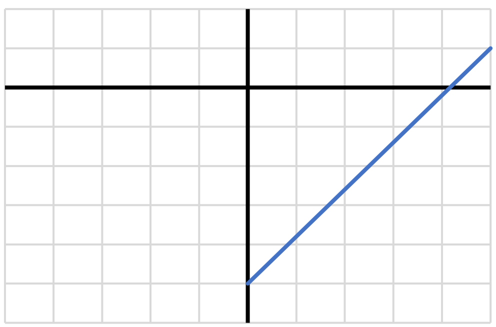

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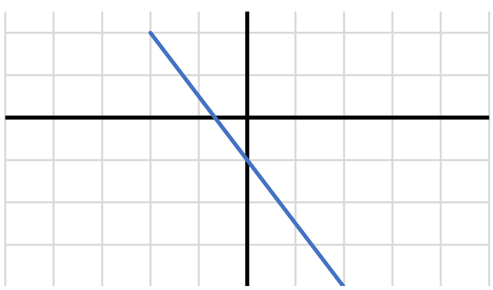

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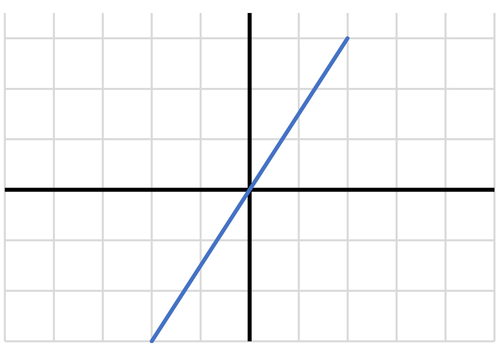

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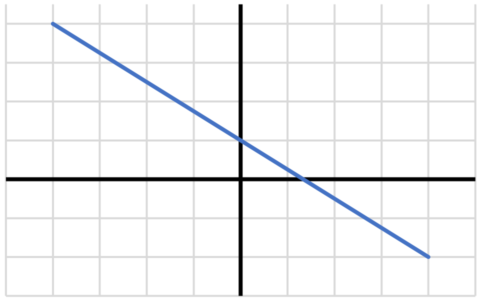

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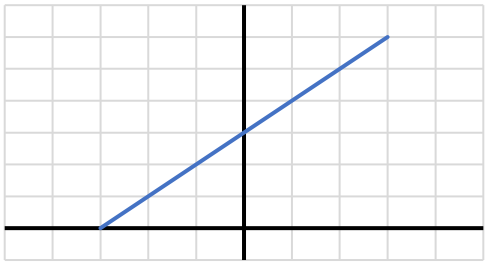

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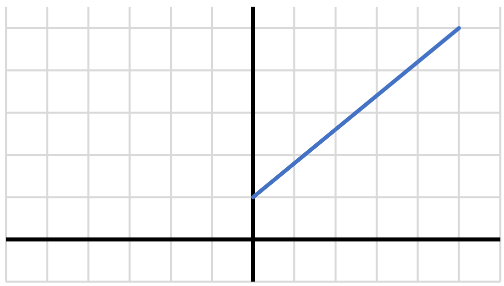

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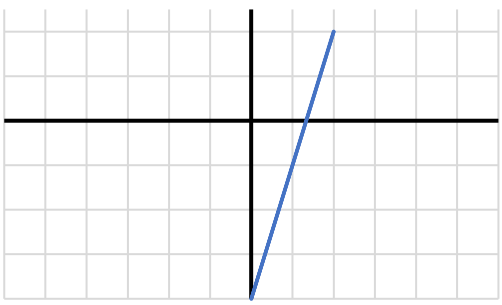

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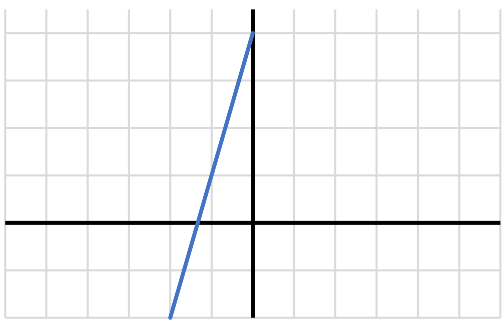

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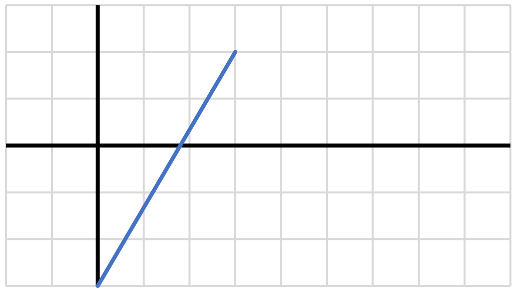

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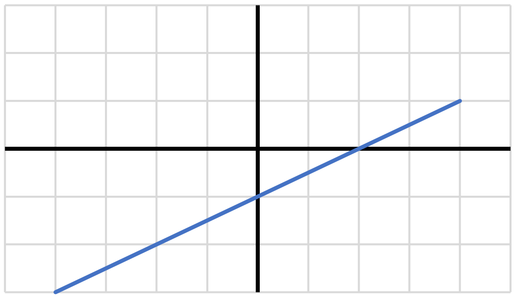


41) $y = 70 - 5x$ with slope -5 and *y*-intercept $(0, 70)$. The slope represents the average rate of change. As the hour increases by 1, the temperature decreases by 5. The *y*-intercept represents the temperature at 0 hours, or the initial temperature.

42) $y = 2.95x + 4.95$ with slope 2.95 and *y*-intercept $(0, 4.95)$. The slope represents the average rate of change. As the month increases by 1, the total cost increases by 2.95. The *y*-intercept represents the total cost at 0 months, or the initial cost.

Section 1.5: Point-Slope Form

Objective: Give the equation of a line with a known slope and point.

The slope-intercept form has the advantage of being simple to remember and use, however, it has one major disadvantage: we must know the $y-$intercept in order to use it! Generally we do not know the $y-$intercept; we only know one or more points (that are not the $y-$intercept). In these cases we can't (easily) use the slope-intercept equation, so we will use a different more flexible formula. If we let the slope of a line be m, and a specific point on the line be (x_1, y_1), and any other point on the line be (x, y), then we can use the point-slope formula to make a second equation.

Example 1.

$m, (x_1, y_1), (x, y)$	Recall slope formula
$\dfrac{y_2 - y_1}{x_2 - x_1} = m$	Plug in values
$\dfrac{y - y_1}{x - x_1} = m$	Multiply both sides by $(x - x_1)$
$y - y_1 = m(x - x_1)$	Our Solution

If we know the slope, m of a line and any point on the line (x_1, y_1) we can easily plug these values into the equation above which will be called the point-slope formula.

Point – Slope Formula: $\boldsymbol{y - y_1 = m(x - x_1)}$

Example 2.

Write the equation of the line through the point $(3, -4)$ with a slope of $\frac{3}{5}$ in point-slope form.

$y - y_1 = m(x - x_1)$	Plug values into point-slope formula
$y - (-4) = \dfrac{3}{5}(x - 3)$	Simplify signs
$y + 4 = \dfrac{3}{5}(x - 3)$	Our Solution

Often, we will prefer final answers be written in slope-intercept form. If the directions ask for the answer in slope-intercept form we will simply distribute the slope, then solve for y.

Example 3.

Write the equation of the line through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

$y - y_1 = m(x - x_1)$	Plug values into point-formula
$y - 2 = -\frac{2}{3}\left(x - (-6)\right)$	Simplify signs
$y - 2 = -\frac{2}{3}(x + 6)$	Distribute slope
$y - 2 = -\frac{2}{3}x - 4$ $+2 \qquad\qquad +2$ $y = -\frac{2}{3}x - 2$	Our Solution

An important thing to observe about the point-slope formula is that the operation between the x's and y's is subtraction. This means when you simplify the signs you will have the opposite of the numbers in the point. We need to be very careful with signs as we use the point-slope formula.

In order to find the equation of a line we will always need to know the slope. If we don't know the slope to begin with we will have to do some work to find it first before we can get an equation.

Example 4.

Find the equation of the line through the points $(-2,5)$ and $(4,-3)$ in point-slope form.

$m = \frac{y_2 - y_1}{x_2 - x_1}$	First we must find the slope
$m = \frac{-3-5}{4-(-2)} = \frac{-8}{6} = -\frac{4}{3}$	Plug values in slope formula and evaluate
$y - y_1 = m(x - x_1)$	With slope and either point, use point-slope formula
$y - 5 = -\frac{4}{3}(x - (-2))$	Simplify signs
$y - 5 = -\frac{4}{3}(x + 2)$	Our Solution

Example 5.

Find the equation of the line through the points $(-3, 4)$ and $(-1, -2)$ in slope-intercept form.

$m = \frac{y_2 - y_1}{x_2 - x_1}$	First we must find the slope
$m = \frac{-2-4}{-1-(-3)} = \frac{-6}{2} = -3$	Plug values in slope formula and evaluate
$y - y_1 = m(x - x_1)$	With slope and either point, use point-slope formula
$y - 4 = -3(x - (-3))$	Simplify signs
$y - 4 = -3(x + 3)$	Distribute slope
$y - 4 = -3x - 9$ $+4 \quad\quad +4$ $y = -3x - 5$	Solve for *y* Add 4 to both sides Our Solution

Example 6.

Find the equation of the line through the points $(6, -2)$ and $(-4, 1)$ in slope-intercept form.

$m = \frac{y_2 - y_1}{x_2 - x_1}$	First we must find the slope
$m = \frac{1-(-2)}{-4-6} = \frac{3}{-10} = -\frac{3}{10}$	Plug values in slope formula and evaluate
$y - y_1 = m(x - x_1)$	Use slope and either point, use point-slope formula
$y - (-2) = -\frac{3}{10}(x - 6)$	Simplify signs
$y + 2 = -\frac{3}{10}(x - 6)$	Distribute slope
$y + 2 = -\frac{3}{10}x + \frac{9}{5}$ $-2 \quad\quad -\frac{10}{5}$ $y = -\frac{3}{10}x - \frac{1}{5}$	Solve for *y*. Subtract 2 form both sides Using $\frac{10}{5}$ on right so we have a common denominator Our Solution

Note: When the slope is undefined or both points have the same x-coordinate, there is no value for *m* because the line is vertical. That is, all of the x-coordinates of the points on that line have the same value, which is the x-coordinate of the given point or points. Therefore, the equation of the line is simply the value of the x-coordinate of the given point or points.

1.5 Practice

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through $(2,2)$, slope $= \frac{1}{2}$
2) through $(2,1)$, slope $= -\frac{1}{2}$
3) through $(-1,-5)$, slope $= 9$
4) through $(2,-2)$, slope $= -2$
5) through $(-4,1)$, slope $= \frac{3}{4}$
6) through $(4,-3)$, slope $= -2$
7) through $(0,-2)$, slope $= -3$
8) through $(-1,1)$, slope $= 4$
9) through $(0,-5)$, slope $= -\frac{1}{4}$
10) through $(0,2)$, slope $= -\frac{5}{4}$
11) through $(-5,-3)$, slope $= \frac{1}{5}$
12) through $(-1,-4)$, slope $= -\frac{2}{3}$
13) through $(-1,4)$, slope $= -\frac{5}{4}$
14) through $(1,-4)$, slope $= -\frac{3}{2}$

Write the slope-intercept form of the equation of the line through the given point with the given slope.

15) through: $(-1,-5)$, slope $= 2$
16) through: $(2,-2)$, slope $= -2$
17) through: $(5,-1)$, slope $= -\frac{3}{5}$
18) through: $(-2,-2)$, slope $= -\frac{2}{3}$
19) through: $(-4,1)$, slope $= \frac{1}{2}$
20) through: $(4,-3)$, slope $= -\frac{7}{4}$
21) through: $(4,-2)$, slope $= -\frac{3}{2}$
22) through: $(-2,0)$, slope $= -\frac{5}{2}$
23) through: $(-5,-3)$, slope $= -\frac{2}{5}$
24) through: $(3,3)$, slope $= \frac{7}{3}$
25) through: $(2,-2)$, slope $= 1$

26) through: $(-4,-3)$, slope $= 0$

27) through: $(-2,-5)$, slope $= 2$

28) through: $(-4, 2)$, slope $= -\frac{1}{2}$

29) through: $(5, 3)$, slope $= \frac{6}{5}$

Write the point-slope form of the equation of the line through the given points.

30) through: $(-4, 3)$ and $(-3, 1)$

31) through: $(1, 3)$ and $(-3, 3)$

32) through: $(5, 1)$ and $(-3, 0)$

33) through: $(-4, 5)$ and $(4, 4)$

34) through: $(-4,-2)$ and $(0, 4)$

35) through: $(-4, 1)$ and $(4, 4)$

36) through: $(3, 5)$ and $(-5, 3)$

37) through: $(-1,-4)$ and $(-5, 0)$

38) through: $(3,-3)$ and $(-4, 5)$

39) through: $(-1,-5)$ and $(-5,-4)$

Write the slope-intercept form of the equation of the line through the given points.

40) through: $(-5, 1)$ and $(-1,-2)$

41) through: $(-5,-1)$ and $(5,-2)$

42) through: $(-5, 5)$ and $(2,-3)$

43) through: $(1,-1)$ and $(-5,-4)$

44) through: $(4, 1)$ and $(1, 4)$

45) through: $(0, 1)$ and $(-3, 0)$

46) through: $(0, 2)$ and $(5,-3)$

47) through: $(0, 2)$ and $(2, 4)$

48) through: $(0, 3)$ and $(-1,-1)$

49) through: $(-2, 0)$ and $(5, 3)$

1.5 Answers

1) $y-2=\frac{1}{2}(x-2)$
2) $y-1=-\frac{1}{2}(x-2)$
3) $y+5=9(x+1)$
4) $y+2=-2(x-2)$
5) $y-1=\frac{3}{4}(x+4)$
6) $y+3=-2(x-4)$
7) $y+2=-3x$
8) $y-1=4(x+1)$
9) $y+5=-\frac{1}{4}x$
10) $y-2=-\frac{5}{4}x$
11) $y+3=\frac{1}{5}(x+5)$
12) $y+4=-\frac{2}{3}(x+1)$
13) $y-4=-\frac{5}{4}(x+1)$
14) $y+4=-\frac{3}{2}(x-1)$
15) $y=2x-3$
16) $y=-2x+2$
17) $y=-\frac{3}{5}x+2$
18) $y=-\frac{2}{3}x-\frac{10}{3}$
19) $y=\frac{1}{2}x+3$
20) $y=-\frac{7}{4}x+4$
21) $y=-\frac{3}{2}x+4$
22) $y=-\frac{5}{2}x-5$
23) $y=-\frac{2}{5}x-5$
24) $y=\frac{7}{3}x-4$
25) $y=x-4$
26) $y=-3$
27) $y=2x-1$
28) $y=-\frac{1}{2}x$
29) $y=\frac{6}{5}x-3$
30) $y-3=-2(x+4)$

31) $y=3$

32) $y-1=\frac{1}{8}(x-5)$

33) $y-5=-\frac{1}{8}(x+4)$

34) $y+2=\frac{3}{2}(x+4)$

35) $y-1=\frac{3}{8}(x+4)$

36) $y-5=\frac{1}{4}(x-3)$

37) $y+4=-(x+1)$

38) $y+3=-\frac{8}{7}(x-3)$

39) $y+5=-\frac{1}{4}(x+1)$

40) $y=-\frac{3}{4}x-\frac{11}{4}$

41) $y=-\frac{1}{10}x-\frac{3}{2}$

42) $y=-\frac{8}{7}x-\frac{5}{7}$

43) $y=\frac{1}{2}x-\frac{3}{2}$

44) $y=-x+5$

45) $y=\frac{1}{3}x+1$

46) $y=-x+2$

47) $y=x+2$

48) $y=4x+3$

49) $y=\frac{3}{7}x+\frac{6}{7}$

Chapter 2: Systems of Equations

Section 2.1: Solving Systems of Equations by Graphing

Objective: Solve systems of equations by graphing and identifying the point of intersection.

We have solved equations like $3x-4=11$ by adding 4 to both sides and then dividing by 3 (solution is $x=5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have several equations we are using to solve, we call the equations a **system of equations**. When solving a system of equations we are looking for a solution that works for all of these equations. In this discussion, we will limit ourselves to solving two equations with two unknowns. This solution is usually given as an ordered pair (x, y). The following example illustrates a solution working in both equations.

Example 1.

Show $(2,1)$ is the solution to the system $\begin{aligned} 3x-y&=5 \\ x+y&=3 \end{aligned}$

$(2,1)$	Identify x and y from the ordered pair
$x=2,\ y=1$	Plug these values into each equation
$3(2)-(1)=5$	First equation
$6-1=5$	Evaluate
$5=5$	True
$(2)+(1)=3$	Second equation, evaluate
$3=3$	True

As we found a true statement for both equations we know $(2,1)$ is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this lesson we will be working to find this point given the equations. It seems to follow that if we use points to describe the solution, we can use graphs to find the solutions.

If the graph of a line is a picture of all the solutions, we can graph two lines on the same coordinate plane to see the solutions of both equations. We are interested in the point that is a solution for both lines, this would be where the lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 2.

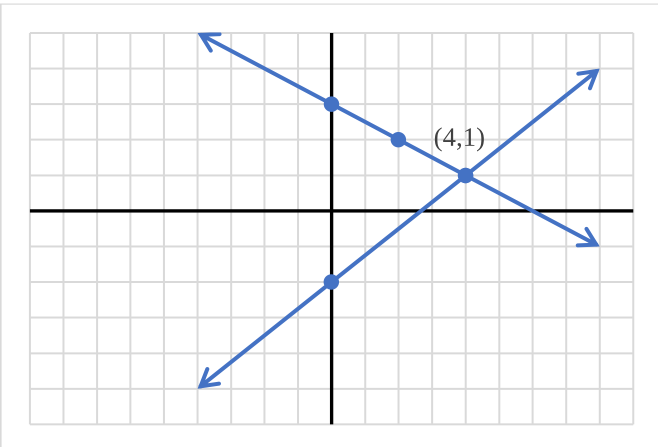
Solve the system of equations by graphing: $\begin{array}{l} y=-\frac{1}{2}x+3 \\ y=\frac{3}{4}x-2 \end{array}$

$$y=-\frac{1}{2}x+3 \quad \text{To graph we identify slopes and } y\text{-intercepts}$$

$$y=\frac{3}{4}x-2$$

$$\text{First: } m=-\frac{1}{2},\ b=3$$

$$\text{Second: } m=\frac{3}{4},\ b=-2 \quad \text{Now we can graph both lines on the same plane.}$$



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, $(4,1)$

$(4,1)$ Our Solution

Often the equations will not be in slope-intercept form. We can solve both equations for y first to put the equation in slope-intercept form.

Example 3.

Solve the system of equations by graphing: $\begin{array}{l} 6x-3y=-9 \\ 2x+2y=-6 \end{array}$

$$\begin{array}{l} 6x-3y=-9 \\ 2x+2y=-6 \end{array} \quad \text{Solve each equation for } y$$

$$\begin{array}{rr} 6x-3y=-9 & 2x+2y=-6 \\ \underline{-6x \qquad -6x} & \underline{-2x \qquad -2x} \\ -3y=-6x-9 & 2y=-2x-6 \end{array} \quad \begin{array}{l} \text{Subtract } x \text{ terms} \\ \text{Put } x \text{ terms first} \end{array}$$

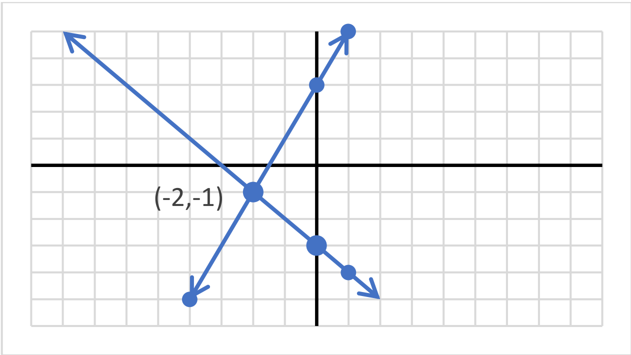
$$\frac{-3y}{-3}=\frac{-6x}{-3}-\frac{9}{-3} \qquad \frac{2y}{2}=\frac{-2x}{2}-\frac{6}{2} \quad \text{Divide by coefficient of } y$$

$y=2x+3 \qquad y=-x-3$ Identify slope and y-intercepts

First: $m=\frac{2}{1}, b=3$

Second: $m=-\frac{1}{1}, b=-3$

Now we can graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, $(-2,-1)$

$(-2,-1)$ Our Solution

As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two examples.

Example 4.

Solve the system of equations by graphing: $y=\frac{3}{2}x-4$, $y=\frac{3}{2}x+1$

$y=\frac{3}{2}x-4$

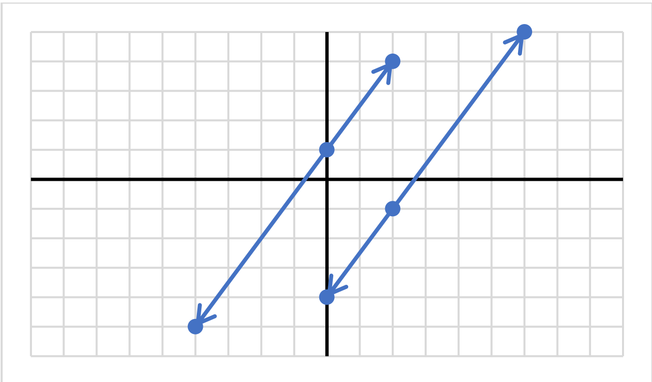
$y=\frac{3}{2}x+1$

Identify slope and y-intercept of each equation

First: $m=\frac{3}{2}, b=-4$

Second: $m=\frac{3}{2}, b=1$

Now we can graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

The two lines do not intersect! They are parallel! If the lines do not intersect we know that there is no point that works in both equations. There is no solution.

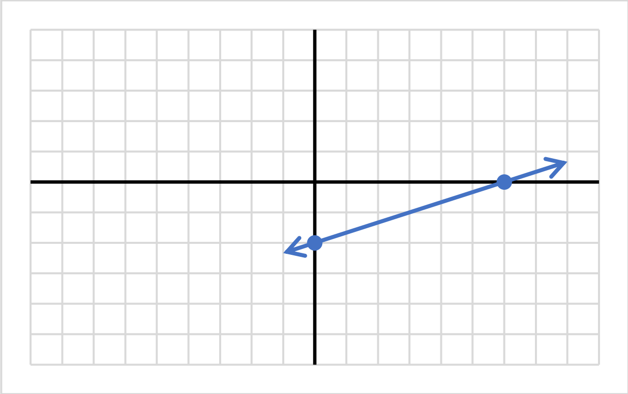
$\varnothing$ No Solution

You can graph lines using intercepts as well.

Example 5.

Solve the system of equations by graphing: $\begin{aligned} 2x - 6y &= 12 \\ 3x - 9y &= 18 \end{aligned}$

$2x - 6y = 12$	Let us graph using intercepts.
$2x - 6(0) = 12$	x-intercept (let $y = 0$)
$2x = 12$	
$x = 6$	
$(6, 0)$	Our x-intercept
$2(0) - 6y = 12$	
$-6y = 12$	
$y = -2$	
$(0, -2)$	Our y-intercept
$3x - 9y = 18$	Let us graph using intercepts again.
$3x - 9(0) = 18$	x-intercept (let $y = 0$)
$3x = 18$	
$x = 6$	
$(6, 0)$	Our x-intercept
$3(0) - 9y = 18$	
$-9y = 18$	
$y = -2$	
$(0, -2)$	Our y-intercept



Both equations are the same line! As one line is directly on top of the other line, we can say that the lines "intersect" at all the points! Here we say we have infinite solutions.

World View Note: The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks who would solve systems of equations with three or four variables and, around 300 AD, developed methods for solving systems with any number of unknowns!

2.1 Practice

Solve each system of equations by graphing.

1) $y = -x + 1$
$y = -5x - 3$

2) $y = -\frac{5}{4}x - 2$
$y = -\frac{1}{4}x + 2$

3) $y = -3$
$y = -x - 4$

4) $y = -x - 2$
$y = \frac{2}{3}x + 3$

5) $y = -\frac{3}{4}x + 1$
$y = -\frac{3}{4}x + 2$

6) $y = 2x + 2$
$y = -x - 4$

7) $y = \frac{1}{3}x + 2$
$y = -\frac{5}{3}x - 4$

8) $y = 2x - 4$
$y = \frac{1}{2}x + 2$

9) $y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

10) $y = \frac{1}{2}x + 4$
$y = \frac{1}{2}x + 1$

11) $x + 3y = -9$
$5x + 3y = 3$

12) $x + 4y = -12$
$2x + y = 4$

13) $\begin{aligned} x - y &= 4 \\ 2x + y &= -1 \end{aligned}$

14) $\begin{aligned} 6x + y &= -3 \\ x + y &= 2 \end{aligned}$

15) $\begin{aligned} 2x + 3y &= -6 \\ 2x + y &= 2 \end{aligned}$

16) $\begin{aligned} 3x + 2y &= 2 \\ 3x + 2y &= -6 \end{aligned}$

17) $\begin{aligned} 2x + y &= 2 \\ x - y &= 4 \end{aligned}$

18) $\begin{aligned} x + 2y &= 6 \\ 5x - 4y &= 16 \end{aligned}$

19) $\begin{aligned} 2x + y &= -2 \\ x + 3y &= 9 \end{aligned}$

20) $\begin{aligned} x - y &= 3 \\ 5x + 2y &= 8 \end{aligned}$

21) $\begin{aligned} 2y &= 4x + 6 \\ y &= 2x + 3 \end{aligned}$

22) $\begin{aligned} -2y + x &= 4 \\ 2 &= -x + \tfrac{1}{2}y \end{aligned}$

23) $\begin{aligned} 2x - y &= -1 \\ 0 &= -2x - y - 3 \end{aligned}$

24) $\begin{aligned} -2y &= -4 - x \\ -2y &= -5x + 4 \end{aligned}$

25) $\begin{aligned} 3 + y &= -x \\ -4 - 6x &= -y \end{aligned}$

26) $\begin{aligned} 16 &= -x - 4y \\ -2x &= -4 - 4y \end{aligned}$

27) $\begin{aligned} -y + 7x &= 4 \\ -y - 3 + 7x &= 0 \end{aligned}$

28) $\begin{array}{l} -4 + y = x \\ x + 2 = -y \end{array}$

29) $\begin{array}{l} -12 + x = 4y \\ 12 - 5x = 4y \end{array}$

30) $\begin{array}{l} 12x - 3y = 9 \\ y = 4x - 3 \end{array}$

2.1 Answers

1) $(-1, 2)$
2) $(-4, 3)$
3) $(-1, -3)$
4) $(-3, 1)$
5) No Solution
6) $(-2, -2)$
7) $(-3, 1)$
8) $(4, 4)$
9) $(-3, -1)$
10) No Solution
11) $(3, -4)$
12) $(4, -4)$
13) $(1, -3)$
14) $(-1, 3)$
15) $(3, -4)$
16) No Solution
17) $(2, -2)$
18) $(4, 1)$
19) $(-3, 4)$
20) $(2, -1)$
21) Infinite Solutions
22) $(-4, -4)$
23) $(-1, -1)$
24) $(2, 3)$
25) $(-1, -2)$
26) $(-4, -3)$
27) No Solution
28) $(-3, 1)$
29) $(4, -2)$
30) Infinite Solutions

Section 2.2: Solving Systems of Equations by Substitution

Objective: Solve systems of equations using substitution.

Solving a system by graphing has several limitations. First, it requires the graph to be perfectly drawn**.** If the lines are not straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, such as $(100,-75)$, or if the answer contains a decimal that the graph will not help us find, such as $\left(3.2134,\ 2.17\right)$. For these reasons we will rarely use graphing to solve our systems. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several examples, then end with a five-step process to solve problems using this method.

Example 1.

Solve the systems of equations by using substitution: $\begin{aligned} x &= 5 \\ y &= 2x-3 \end{aligned}$

$x=5$ $y=2x-3$	We already know $x=5$, substitute this into the other equation
$y=2(\mathbf{5})-3$	Evaluate, multiply first
$y=10-3$	Subtract
$y=7$	We now also have y
$(5,7)$	Our Solution

When we know what one variable equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parentheses. The reason for this is shown in the next example.

Example 2.

Solve the systems of equations by using substitution: $\begin{aligned} 2x-3y &= 7 \\ y &= 3x-7 \end{aligned}$

$2x-3y=7$ $y=3x-7$	We know $y=3x-7$; substitute this into the other equation
$2x-3(\mathbf{3x-7})=7$	Solve this equation, distributing -3 first
$2x-9x+21=7$	Combine like terms $2x-9x$
$-7x+21=7$	Subtract 21

$$\begin{array}{rl}
-7x+21=7 & \\
\mathbf{-21\quad -21} & \\
\hline
-7x=-14 & \\
\dfrac{-7x}{\mathbf{-7}}=\dfrac{-14}{\mathbf{-7}} & \text{Divide by } -7 \\
x=2 & \text{We now have our } x\text{; plug into the } y= \text{ equation to find } y \\
y=3(\mathbf{2})-7 & \text{Evaluate, multiply first} \\
y=6-7 & \text{Subtract} \\
y=-1 & \text{We now also have } y \\
(2,-1) & \text{Our Solution}
\end{array}$$

By using the entire expression $3x-7$ to replace y in the other equation we were able to reduce the system to a single linear equation, which we can easily solve for our first variable. However, the lone variable (a variable without a coefficient) is not always alone on one side of the equation. If this happens we can isolate it by solving for the lone variable.

Example 3.

Solve the systems of equations by using substitution: $\begin{array}{l} 3x+2y=1 \\ x-5y=6 \end{array}$

$$\begin{array}{rl}
3x+2y=1 & \text{Lone variable is } x\text{; isolate by adding } 5y \text{ to both sides.} \\
x-5y=6 & \\
\mathbf{+5y\quad +5y} & \\
\hline
x=6+5y & \text{Substitute this into the untouched equation} \\
3(\mathbf{6+5y})+2y=1 & \text{Solve this equation, distributing } 3 \text{ first} \\
18+15y+2y=1 & \text{Combine like terms } 15y+2y \\
18+17y=1 & \\
\mathbf{-18\qquad -18} & \\
\hline
17y=-17 & \\
\dfrac{17y}{\mathbf{17}}=\dfrac{-17}{\mathbf{17}} & \text{Divide both sides by } 17 \\
y=-1 & \text{We have our } y\text{; plug this into the } x= \text{ equation to find } x \\
x=6+5(-1) & \text{Evaluate, multiply first} \\
x=6-5 & \text{Subtract} \\
x=1 & \text{We now also have } x \\
(1,-1) & \text{Our Solution}
\end{array}$$

The process in the previous example is how we will solve problems using substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x-2y=2$ $2x+y=-5$
1. Find the lone variable	Second Equation, y $2x+\boldsymbol{y}=-5$
2. Solve for the lone variable	$-2x \quad -2x$ $\boldsymbol{y=-5-2x}$
3. Substitute into the untouched equation	$4x-2(\boldsymbol{-5-2x})=2$
4. Solve	$4x+10+4x=2$ $8x+10=2$ $-10 \quad -10$ $\frac{8x}{8}=\frac{-8}{8}$ $x=-1$
5. Plug into lone variable equation and evaluate	$y=-5-2(\boldsymbol{-1})$ $y=-5+2$ $\boldsymbol{y=-3}$
Solution	$(-1,-3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for; either choice will give the same final result.

Example 4.

Solve the system of equations by using substitution.

$\boldsymbol{x}+y=5$	Find the lone variable: x or y in first, or x in second.
$x-y=-1$	We will chose x in the first
$x+y=5$ $\boldsymbol{-y \quad -y}$	Solve for the lone variable; subtract y from both sides
$x=5-y$	Plug into the untouched equation, the second equation
$\boldsymbol{(5-y)}-y=-1$	Solve (parentheses are not needed here); combine like terms
$5-2y=-1$ $-5 \quad -5$	Subtract 5 from both sides
$-2y=-6$	
$\frac{-2y}{-2}=\frac{-6}{-2}$	Divide both sides by -2

$y=3$	We have our y!
$x=5-(3)$	Plug into lone variable equation; evaluate
$x=2$	Now we have our x
$(2,3)$	Our Solution

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 5.

$\boldsymbol{y}+4=3x$	Find the lone variable, y, in the first equation
$2y-6x=-8$	
$y+4=3x$	Solve for the lone variable; subtract 4 from both sides
$\underline{\;-4\quad -4}$	
$y=3x-4$	Plug into untouched equation
$2(\boldsymbol{3x-4})-6x=-8$	Solve; first distribute 2 through parentheses grouping
$6x-8-6x=-8$	Combine like terms $6x-6x$
$-8=-8$	Variables are gone! A true statement.
Infinite solutions	Our Solution

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 6.

$6x-3y=-9$	Find the lone variable, y, in the second equation
$-2x+\boldsymbol{y}=5$	
$-2x+y=5$	Solve for the lone variable; add $2x$ to both sides
$\underline{+2x\qquad +2x}$	
$y=5+2x$	Plug into untouched equation
$6x-3(\boldsymbol{5+2x})=-9$	Solve, first distribute through parentheses grouping
$6x-15-6x=-9$	Combine like terms $6x-6x$
$-15\neq-9$	Variables are gone! A false statement.
No solution $\varnothing$	Our Solution

Because we had a false statement and no variables, we know that no numerical values will work in both equations.

World View Note: French mathematician Rene Descartes wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is why often we are solving for the variables x, y and z.

One more question needs to be considered, what if there is no lone variable? If there is no lone variable substitution can still work to solve, we will just have to select one variable to solve for and use fractions as we solve.

Example 7.

$5x-6y=-14$	No lone variable
$-2x+4y=12$	We will solve for x in the first equation
$5x-6y=-14$	Solve for our variable, add $6y$ to both sides
$\underline{\quad +\mathbf{6y} \quad +\mathbf{6y}}$	
$\frac{5x}{\mathbf{5}}=\frac{-14}{\mathbf{5}}+\frac{6y}{\mathbf{5}}$	Divide each term by 5
$x=\frac{-14}{5}+\frac{6y}{5}$	Plug into untouched equation
$-2\left(\frac{-14}{5}+\frac{6y}{5}\right)+4y=12$	Solve, distribute through parenthesis
$\frac{28}{5}-\frac{12y}{5}+4y=12$	Clear fractions by multiplying by 5
$\frac{28(\mathbf{5})}{5}-\frac{12y(\mathbf{5})}{5}+4y(\mathbf{5})=12(\mathbf{5})$	Reduce fractions and multiply
$28-12y+20y=60$	Combine like terms $-12y+20y$
$28+8y=60$	Subtract 28 from both sides
$\underline{\mathbf{-28} \quad\quad \mathbf{-28}}$	
$\frac{8y}{\mathbf{8}}=\frac{32}{\mathbf{8}}$	Divide both sides by 8
$y=4$	We have our y
$x=\frac{-14}{5}+\frac{6(4)}{5}$	Plug into lone variable equation, multiply
$x=\frac{-14}{5}+\frac{24}{5}$	Add fractions
$x=\frac{10}{5}$	Reduce fraction
$x=2$	Now we have our x
$(2,4)$	Our Solution

Using the fractions could make the problem a bit trickier. This is why we have another method for solving systems of equations that will be discussed in another lesson.

2.2 Practice

Solve each system by substitution.

1) $y = -3x$
$y = 6x - 9$

2) $y = x + 5$
$y = -2x - 4$

3) $y = -2x - 9$
$y = 2x - 1$

4) $y = -6x + 3$
$y = 6x + 3$

5) $y = 6x + 4$
$y = -3x - 5$

6) $y = 3x + 13$
$y = -2x - 22$

7) $y = 3x + 2$
$y = -3x + 8$

8) $y = -2x - 9$
$y = -5x - 21$

9) $y = 2x - 3$
$y = -2x + 9$

10) $y = 7x - 24$
$y = -3x + 16$

11) $y = 6x - 6$
$-3x - 3y = -24$

12) $-x + 3y = 12$
$y = 6x + 21$

13) $y = -6$
$3x - 6y = 30$

14) $6x - 4y = -8$
$y = -6x + 2$

15) $\begin{array}{l} y=-5 \\ 3x+4y=-17 \end{array}$

16) $\begin{array}{l} 7x+2y=-7 \\ y=5x+5 \end{array}$

17) $\begin{array}{l} -2x+2y=18 \\ y=7x+15 \end{array}$

18) $\begin{array}{l} y=x+4 \\ 3x-4y=-19 \end{array}$

19) $\begin{array}{l} y=-8x+19 \\ -x+6y=16 \end{array}$

20) $\begin{array}{l} y=-2x+8 \\ -7x-6y=-8 \end{array}$

21) $\begin{array}{l} 7x-3y=-7 \\ y=4x+1 \end{array}$

22) $\begin{array}{l} 3x-2y=-13 \\ 4x+2y=18 \end{array}$

23) $\begin{array}{l} x-5y=7 \\ 2x+7y=-20 \end{array}$

24) $\begin{array}{l} 3x-4y=15 \\ 7x+y=4 \end{array}$

25) $\begin{array}{l} -2x-5y=-5 \\ x-8y=-23 \end{array}$

26) $\begin{array}{l} 6x+4y=16 \\ -2x+9y=-3 \end{array}$

27) $\begin{array}{l} -6x+y=20 \\ -3x-3y=-18 \end{array}$

28) $\begin{array}{l} 7x+5y=-13 \\ x-4y=-16 \end{array}$

29) $\begin{array}{l} 3x+y=9 \\ 2x+8y=-16 \end{array}$

30) $\begin{array}{l} -5x-5y=-20 \\ -2x+y=7 \end{array}$

31) $\begin{array}{l} 2x+y=2 \\ 3x+7y=14 \end{array}$

32) $\begin{array}{l} x=3y+4 \\ 2x-6y=8 \end{array}$

33) $\begin{array}{l} 4x-y=6 \\ y=4x-3 \end{array}$

34) $\begin{array}{l} y=x+7 \\ x-y=3 \end{array}$

35) $\begin{array}{l} -9x+6y=21 \\ 2y-7=3x \end{array}$

36) $\begin{array}{l} y=3x+5 \\ y=3x-1 \end{array}$

37) $\begin{array}{l} 3x-3y=-24 \\ x-y=5 \end{array}$

38) $\begin{array}{l} x+2y=1 \\ 5x=5-10y \end{array}$

39) $\begin{array}{l} 12x+6y=0 \\ y=-2x \end{array}$

40) $\begin{array}{l} -x-4y=-14 \\ -6x+8y=12 \end{array}$

2.2 Answers

1) $(1,-3)$
2) $(-3,2)$
3) $(-2,-5)$
4) $(0,3)$
5) $(-1,-2)$
6) $(-7,-8)$
7) $(1,5)$
8) $(-4,-1)$
9) $(3,3)$
10) $(4,4)$
11) $(2,6)$
12) $(-3,3)$
13) $(-2,-6)$
14) $(0,2)$
15) $(1,-5)$
16) $(-1,0)$
17) $(-1,8)$
18) $(3,7)$
19) $(2,3)$
20) $(8,-8)$
21) $\left(\frac{4}{5},\frac{21}{5}\right)$
22) $\left(\frac{5}{7},\frac{53}{7}\right)$
23) $(-3,-2)$
24) $(1,-3)$
25) $\left(-\frac{25}{7},\frac{17}{7}\right)$
26) $\left(\frac{78}{31},\frac{7}{31}\right)$
27) $(-2,8)$
28) $(-4,3)$
29) $(4,-3)$
30) $(-1,5)$
31) $(0,2)$
32) Infinite Solutions
33) No Solution

34) No Solution
35) Infinite Solutions
36) No Solution
37) No Solution
38) Infinite Solutions
39) Infinite Solutions
40) $(2, 3)$

Section 2.3: Solving Systems of Equations by Addition

Objective: Solve systems of equations using the addition method.

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as substitution. This is probably the most used idea in solving systems in various areas of algebra. However, substitution can get ugly if we don't have a lone variable. This leads us to our third method for solving systems of equations: the **addition method** (sometimes called the elimination method). We will set up the process in the following examples, then define the five step process we can use to solve by addition.

Example 1.

Solve the systems of equations by addition: $\begin{array}{l} 3x-4y=8 \\ 5x+4y=-24 \end{array}$

$3x-4y=8$ $\underline{5x+4y=-24}$	Notice opposites in front of y's. Add columns.
$8x \quad = -16$	
$\dfrac{8x}{8} = \dfrac{-16}{8}$	Solve for x, divide by 8
$x=-2$	We have our x!
$5(-2)+4y=-24$	Plug into either original equation, simplify
$-10+4y=-24$ $\underline{+10 \qquad +10}$	Add 10 to both sides
$\dfrac{4y}{4}=\dfrac{-14}{4}$	Divide by 4
$y=-\dfrac{7}{2}$	Now we have our y!
$\left(-2,-\dfrac{7}{2}\right)$	Our Solution

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 2.

Solve the systems of equations by addition: $\begin{aligned}-6x+5y&=22\\2x+3y&=2\end{aligned}$

$-6x+5y=22$ $2x+3y=2$	We can get opposites in front of x, by multiplying the second equation by 3, to get $-6x$ and $+6x$
$3(2+3y)=(2)3$	Distribute to get the new second equation.
$6x+9y=6$	First equation still the same, add
$\underline{-6x+5y=22}$	
$14y=28$	
$\frac{14y}{14}=\frac{28}{14}$	Divide both sides by 14
$y=2$	We have our y!
$2x+3(2)=2$	Plug into one of the original equations, simplify
$2x+6=2$	Subtract 6 from both sides
$\underline{\quad-6\quad-6}$	
$\frac{2x}{2}=\frac{-4}{2}$	Divide both sides by 2
$x=-2$	We also have our x!
$(-2,2)$	Our Solution

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by -5 to get opposites, $15y$ and $-15y$.

This illustrates an important point: for some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we want before adding.

Example 3.

Solve the systems of equations by addition: $\begin{aligned}3x+6y&=-9\\2x+9y&=-26\end{aligned}$

$3x+6y=-9$	We can get opposites in front of y, find LCM of 6 and 9
$2x+9y=-26$	The LCM is 18. We will multiply to get $18y$ and $-18y$
$3(3x+6y)=(-9)3$	Multiply the first equation by 3, both sides!
$9x+18y=-27$	

$-2(2x+9y)=(-26)(-2)$	Multiply the second equation by -2, both sides!
$-4x-18y=52$	
$9x+18y=-27$	Add the two new equations together
$\underline{-4x-18y=52}$	
$5x \quad = 25$	Divide both sides by 5
$\frac{5x}{5}=\frac{25}{5}$	
$x=5$	We have our solution for x
$3(5)+6y=-9$	Plug into either original equation; simplify
$15+6y=-9$	Subtract 15 from both sides
$\underline{-15 \qquad -15}$	
$\frac{6y}{6}=\frac{-24}{6}$	Divide both sides by 6
$y=-4$	Now we have our solution for y
$(5,-4)$	Our Solution

It is important for each problem as we get started that all variables and constants are lined up before we start multiplying and adding equations. This is illustrated in the next example, which includes the five steps we will go through to solve a problem using addition.

Problem	$2x-5y=-13$ $-3y+4=-5x$
1. Line up the variables and constants	Second Equation: $-3y+4=-5x$ $+5x-4 \quad +5x-4$ $5x-3y=-4$
2. Multiply to get opposites (use LCM)	$2x-5y=-13$ $5x-3y=-4$ First Equation: multiply by -5 $-5(2x-5y)=(-13)(-5)$ $-10x+25y=65$ Second Equation: multiply by 2 $2(5x-3y)=(-4)2$ $10x-6y=-8$ $-10x+25y=65$ $10x-6y=-8$

3. Add	$19y = 57$
4. Solve	$\frac{19y}{19} = \frac{57}{19}$ $y = 3$
5. Plug into either original and solve	$2x - 5(3) = -13$ $2x - 15 = -13$ $\quad +15 \quad +15$ $\frac{2x}{2} = \frac{2}{2}$ $x = 1$
Solution	$(1, 3)$

World View Note: The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China describes a formula very similar to Gaussian elimination which is very similar to the addition method.

Just as with graphing and substitution, it is possible to have no solution or infinite solutions with addition. Just as with substitution, if the variables all disappear from our problem, a true statement will indicate infinite solutions and a false statement will indicate no solution.

Example 4.

Solve the systems of equations by addition: $\begin{aligned} 2x - 5y &= 3 \\ -6x + 15y &= -9 \end{aligned}$

$2x - 5y = 3$ $-6x + 15y = -9$	To get opposites in front of x, multiply first equation by 3
$3(2x - 5y) = (3)3$ $6x - 15y = 9$	Distribute
$6x - 15y = 9$ $-6x + 15y = -9$	Add equations together
$0 = 0$	True statement
Infinite solutions	Our Solution

Example 5.

Solve the systems of equations by addition: $\begin{aligned} 4x-6y&=8 \\ 6x-9y&=15 \end{aligned}$

$$4x-6y=8 \quad \text{LCM for } x\text{'s is 12}$$
$$6x-9y=15$$

$$3(4x-6y)=(8)3 \quad \text{Multiply first equation by 3}$$
$$12x-18y=24$$

$$-2(6x-9y)=(15)(-2) \quad \text{Multiply second equation by } -2$$
$$-12x+18y=-30$$

$$12x-18y=24 \quad \text{Add both new equations together}$$
$$\underline{-12x+18y=-30}$$
$$0=-6 \quad \text{False statement}$$

No Solution Our Solution

We have covered three different methods that can be used to solve a system of two equations with two variables. While all three can be used to solve any system, graphing works well for small integer solutions. Substitution works well when we have a lone variable, and addition works well when the other two methods fail. As each method has its own strengths, it is important you are familiar with all three methods.

2.3 Practice

Solve each system by the addition (or elimination) method.

1) $\begin{array}{l} 4x+2y=0 \\ -4x-9y=-28 \end{array}$

2) $\begin{array}{l} -7x+y=-10 \\ -9x-y=-22 \end{array}$

3) $\begin{array}{l} -9x+5y=-22 \\ 9x-5y=13 \end{array}$

4) $\begin{array}{l} -x-2y=-7 \\ x+2y=7 \end{array}$

5) $\begin{array}{l} -6x+9y=3 \\ 6x-9y=-9 \end{array}$

6) $\begin{array}{l} 5x-5y=15 \\ -10x+10y=-30 \end{array}$

7) $\begin{array}{l} 4x-6y=-10 \\ 4x-6y=-14 \end{array}$

8) $\begin{array}{l} -3x+3y=-12 \\ -3x+9y=-24 \end{array}$

9) $\begin{array}{l} -x-5y=28 \\ -x+4y=-17 \end{array}$

10) $\begin{array}{l} -10x-5y=0 \\ -10x-10y=-30 \end{array}$

11) $\begin{array}{l} 2x-y=5 \\ 5x+2y=-28 \end{array}$

12) $\begin{array}{l} -5x+6y=-17 \\ x-2y=5 \end{array}$

13) $\begin{array}{l} 10x+6y=24 \\ -6x+y=4 \end{array}$

14) $\begin{array}{l} x+3y=-12 \\ 10x+6y=-10 \end{array}$

15) $\begin{array}{l} 2x+4y=25 \\ 4x-12y=7 \end{array}$

16) $\begin{array}{l} -6x+4y=12 \\ 12x+6y=18 \end{array}$

17) $\begin{array}{l} -7x+4y=-4 \\ 10x-8y=-8 \end{array}$

18) $\begin{array}{l} -6x+4y=4 \\ -3x-y=26 \end{array}$

19) $\begin{array}{l} 5x+10y=20 \\ -6x-5y=-3 \end{array}$

20) $\begin{array}{l} -9x-5y=-19 \\ 3x-7y=-11 \end{array}$

21) $\begin{array}{l} -7x-3y=12 \\ -6x-5y=20 \end{array}$

22) $\begin{array}{l} -5x+4y=4 \\ -7x-10y=-10 \end{array}$

23) $\begin{array}{l} 9x-2y=-18 \\ 5x-7y=-10 \end{array}$

24) $\begin{array}{l} 3x+7y=-8 \\ 4x+6y=-4 \end{array}$

25) $\begin{array}{l} 9x+6y=-21 \\ -10x-9y=28 \end{array}$

26) $\begin{array}{l} -4x-5y=12 \\ -10x+6y=30 \end{array}$

27) $\begin{array}{l} -7x+5y=-18 \\ -3x-3y=12 \end{array}$

28) $\begin{array}{l} 8x+7y=-24 \\ 6x+3y=-18 \end{array}$

29) $\begin{array}{l} -8x-8y=-8 \\ 10x+9y=1 \end{array}$

30) $\begin{array}{l} -7x+10y=13 \\ 4x+9y=22 \end{array}$

31) $\begin{array}{l} 9y=7-x \\ -18y+4x=-26 \end{array}$

32) $\begin{array}{l} 2x+5y=-9 \\ 6x-10y=3 \end{array}$

2.3 Answers

1) $(-2, 4)$
2) $(2, 4)$
3) No solution
4) Infinite number of solutions
5) No solution
6) Infinite number of solutions
7) No solution
8) $(2, -2)$
9) $(-3, -5)$
10) $(-3, 6)$
11) } $(-2, -9)$
12) $(1, -2)$
13) $(0, 4)$
14) $\left(\frac{7}{4}, -\frac{55}{12}\right)$
15) $\left(\frac{41}{5}, \frac{43}{20}\right)$
16) $(0, 3)$
17) $(4, 6)$
18) $(-6, -8)$
19) $(-2, 3)$
20) $(1, 2)$
21) $(0, -4)$
22) $(0, 1)$
23) $(-2, 0)$
24) $(2, -2)$
25) $(-1, -2)$
26) $(-3, 0)$
27) $\left(-\frac{1}{6}, -\frac{23}{6}\right)$
28) $(-3, 0)$
29) $(-8, 9)$
30) $(1, 2)$
31) $(-2, 1)$
32) $\left(-\frac{3}{2}, -\frac{6}{5}\right)$

Section 2.4: Applications of Systems

Objective: Solve application problems by setting up a system of equations.

One application of system of equations are known as value problems. Value problems are ones where each variable has a value attached to it. For example, if our variable is the number of nickels in a person's pocket, those nickels would have a value of five cents each. We will use a table to help us set up and solve value problems. The basic structure of the table is shown below.

	Number	Value	Total
Item 1			
Item 2			
Total			

The first column in the table is used for the number of things we have. Quite often these will be our variables. The second column is used for the value each item has. The third column is used for the total value which we calculate by multiplying the number by the value. For example, if we have 7 dimes, each with a value of 10 cents, the total value is $(7)(10)=70$ cents. The last row of the table is for totals. We only will use the third row (also marked total) for the totals that are given to use. This means sometimes this row may have some blanks in it.

Once the table is filled in we can easily make equations by adding each column, setting it equal to the total at the bottom of the column. This is shown in the following example.

Example 1.

In a child's bank there are 11 coins that have a value of $1.85. The coins are either quarters or dimes. How many coins of each type does the child have?

	Number	Value	Total
Quarter	q	25	
Dime	d	10	
Total			

Using value table; use q for quarters. d for dimes
Each quarter's value is 25 cents, each dime's is 10 cents

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total			

Multiply number by value to get totals

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total	11		185

We have 11 coins total. This is the number total.
We have 1.85 for the final total,
Write final total in cents (185) because 25 and 10 are written using cents

$q+d=11$ $25q+10d=185$	First and last columns are our equations by adding Solve by either addition or substitution. Here we will use the addition method.
$-10(q+d)=(11)(-10)$ $-10q-10d=-110$	Multiply first equation by -10
$-10q-10d=-110$ $\underline{25q+10d=185}$ $15q=75$	Add together equations
$\frac{15q}{15}=\frac{75}{15}$	Divide both sides by 15
$q=5$	We have our q, number of quarters is 5
$(5)+d=11$	Plug into one of original equations
$\underline{-5 \qquad -5}$	Subtract 5 from both sides
$d=6$	We have our d, number of dimes is 6
5 quarters and 6 dimes	Our Solution

World View Note: American coins are the only coins that do not state the value of the coin in numeric form. On the back of the dime it says "one dime" (not 10 cents). On the back of the quarter it says "one quarter" (not 25 cents). On the penny it says "one cent" (not 1 cent). The rest of the world (Euros, Yen, Pesos, etc) all write the value as a number so people who don't speak the language can easily use the coins.

Ticket sales also have a value. Often different types of tickets sell for different prices (values). These problems can be solved in much the same way.

Example 2.

There were 41 tickets sold for an event. Tickets for children cost $1.50 and tickets for adults cost $2.00. Total receipts for the event were $73.50. How many of each type of ticket were sold?

	Number	Value	Total
Child	c	1.5	
Adult	a	2	
Total			

Using our value table, c for child, a for adult
Child tickets have value 1.50; adult value is 2.00
(we can drop the zeros after the decimal point)

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total			

Multiply number by value to get totals

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total	41		73.5

We have 41 tickets sold. This is our number total
The final total was 73.50
Write in dollars as 1.5 and 2 are also dollars

$$c + a = 41$$
$$1.5c + 2a = 73.5$$

First and last columns are our equations by adding
We can solve by either addition or substitution

$c + a = 41$ — We will solve by substitution.

$\underline{-c \qquad -c}$ — Solve for a by subtracting c

$a = 41 - c$

$1.5c + 2(41 - c) = 73.5$ — Substitute into untouched equation

$1.5c + 82 - 2c = 73.5$ — Distribute

$-0.5c + 82 = 73.5$ — Combine like terms

$\underline{\qquad -82 \quad -82}$ — Subtract 82 from both sides

$\dfrac{-0.5c}{-0.5} = \dfrac{-8.5}{-0.5}$ — Divide both sides by -0.5

$c = 17$ — We have c, number of child tickets is 17

$a = 41 - (17)$ — Plug into $a =$ equation to find a

$a = 24$ — We have our a, number of adult tickets is 24

17 child tickets and 24 adult tickets — Our Solution

Some problems will not give us the total number of items we have. Instead they will give a relationship between the items. Here we will have statements such as "There are twice as many dimes as nickels". While it is clear that we need to multiply one variable by 2, it may not be clear which variable gets multiplied by 2.
Generally the equations are backwards from the English sentence. If there are twice as many nickels as dimes, than we multiply the other variable (dimes) by two. So, the equation would be $n = 2d$. This type of problem is in the next example.

Example 3.

A man has a collection of stamps made up of 5¢ stamps and 8¢ stamps. There are three times as many 8¢ stamps as 5¢ stamps. The total value of all the stamps is $3.48. How many of each stamp does he have?

	Number	Value	Total
Five	f	5	
Eight	e	8	
Total			

Use value table, f for five cent stamp, and e for eight
Also list value of each stamp under value column

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			

Multiply number by value to get total

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			348

The final total was 348 (written in cents)
We do not know the total number, this is left blank.

$e = 3f$ **3 times as many 8¢ stamps as 5¢ stamps**

$5f + 8e = 348$ Total column gives second equation

$5f + 8(3f) = 348$ Substitution, substitute first equation in second

$5f + 24f = 348$ Multiply first

$29f = 348$ Combine like terms

$\frac{29f}{29} = \frac{348}{29}$ Divide both sides by 39

$f = 12$ We have f. There are 12 five cent stamps

$e = 3(12)$ Plug into first equation

$e = 36$ We have e, There are 36 eight cent stamps

12 five cent, 36 eight cent stamps Our Solution

The same process for solving value problems can be applied to solving interest problems. Our table titles will be adjusted slightly as we do so.

	Invest	Rate	Interest
Account 1			
Account 2			
Total			

Our first column is for the amount invested in each account. The second column is the interest rate earned (written as a decimal - move decimal point twice left), and the last column is for the amount of interest earned. Just as before, we multiply the investment amount by the rate to find the final column, the interest earned. This is shown in the following example.

Example 4.

A woman invests \$4000 in two accounts, one at 6% interest, the other at 9% interest for one year. At the end of the year she had earned \$270 in interest. How much did she have invested in each account?

	Invest	Rate	Interest
Account 1	x	0.06	
Account 2	y	0.09	
Total			

Use our investment table, x and y for accounts
Fill in interest rates as decimals
Note: 6% = 0.06 and 9% = 0.09

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total			

Multiply across to find interest earned.

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total	4000		270

Total investment is 4000,
Total interest was 270

$$x + y = 4000$$
$$0.06x + 0.09y = 270$$
First and last column give our two equations
Solve by either substitution or addition

$$(-0.06)(x + y) = (4000)(-0.06)$$
$$-0.06x - 0.06y = -240$$
Using addition, multiply the first equation by -0.06

$$\begin{aligned} -0.06x - 0.06y &= -240 \\ 0.06x + 0.09y &= 270 \\ \hline \end{aligned}$$
Add equations together

$$\frac{0.03y}{0.03} = \frac{30}{0.03}$$
Divide both sides by 0.03

$$y = 1000$$
We have y = \$1000 invested at 9%

$$x + 1000 = 4000$$
Plug into original equation

$$\underline{-1000 \quad -1000}$$
Subtract 1000 from both sides

$$x = 3000$$
We have x = \$3000 invested at 6%

\$1000 at 9% and \$3000 at 6% — Our Solution

The same process can be used to find an unknown interest rate.

Example 5.

John invests \$5000 in one account and \$8000 in an account paying 4% more in interest. He earned \$1230 in interest after one year. At what rates did he invest?

	Invest	Rate	Interest
Account 1	5000	x	
Account 2	8000	$x + 0.04$	
Total			

Use our investment table. Use x for first rate
The second rate is 4% higher, or $x + 0.04$
Be sure to write this rate as a decimal!

	Invest	Rate	Interest
Account 1	5000	x	$5000x$
Account 2	8000	$x+0.04$	$8000(x+0.04)$
Total			

Multiply to fill in interest column.
Be sure to distribute $8000(x+0.04)$

	Invest	Rate	Interest
Account 1	5000	x	$5000x$
Account 2	8000	$x+0.04$	$8000x+320$
Total			1230

Total interest was 1230.

$5000x+8000x+320=1230$	Last column gives our equation
$13000x+320=1230$	Combine like terms
$-320 \quad -320$	Subtract 320 from both sides
$\frac{13000x}{13000}=\frac{910}{13000}$	Divide both sides by 13000
$x=0.07$	We have our x, 0.07 or 7% interest
$(0.07)+0.04$	Second account is 4% higher
0.11	The account with \$8000 is at 0.11 or 11% interest
\$5000 at 7%, and \$8000 at 11%	Our Solution

2.4 Practice

Solve.

1) A collection of dimes and quarters is worth \$15.25. There are 103 coins in all. How many of each are there?

2) A collection of half dollars and nickels is worth \$13.40. There are 34 coins in all. How many of each are there?

3) The attendance at a school concert was 578. Admission was \$2.00 for adults and \$1.50 for children. The total receipts were \$985.00. How many adults and how many children attended?

4) A purse contains \$3.90 made up of dimes and quarters. If there are 21 coins in all, how many dimes and how many quarters are there?

5) A boy has \$2.25 in nickels and dimes. If there are twice as many dimes as nickels, how many of each kind does he have?

6) \$3.75 is made up of quarters and half dollars. If the number of quarters exceeds the number of half dollars by 3, how many coins of each denomination are there?

7) A collection of 27 coins consisting of nickels and dimes amounts to \$2.25. How many coins of each kind are there?

8) \$3.25 in dimes and nickels were distributed among 45 boys. If each received one coin, how many received a dime and how many received a nickel?

9) There were 429 people at a play. Admission was \$1 each for adults and 75 cents each for children. The receipts were \$372.50. How many children and how many adults attended?

10) There were 200 tickets sold for a women's basketball game. Tickets for students were 50 cents each and for adults 75 cents each. The total amount of money collected was \$132.50. How many of each type of ticket was sold?

11) There were 203 tickets sold for a volleyball game. For activity-card holders, the price was \$1.25 each and for non-card holders the price was S2 each. The total amount of money collected was \$310. How many of each type of ticket was sold?

12) At a local ball game the hotdogs sold for \$2.50 each and the hamburgers sold for \$2.75 each. There were 131 total sandwiches sold for a total value of \$342. How many of each sandwich was sold?

13) At a recent Vikings game $445 in admission tickets was taken in. The cost of a student ticket was $1.50 and the cost of a non-student ticket was $2.50. A total of 232 tickets were sold. How many students and how many nonstudents attended the game?

14) A bank contains 27 coins in dimes and quarters. The coins have a total value of $4.95. Find the number of dimes and quarters in the bank.

15) A coin purse contains 18 coins in nickels and dimes. The coins have a total value of $1.15. Find the number of nickels and dimes in the coin purse.

16) A business executive bought 40 stamps for $9.60. The purchase included 25¢ stamps and 20¢ stamps. How many of each type of stamp were bought?

17) A postal clerk sold some 15¢ stamps and some 25¢ stamps. Altogether, 15 stamps were sold for a total cost of $3.15. How many of each type of stamp were sold?

18) A drawer contains 15¢ stamps and 18¢ stamps. The number of 15¢ stamps is four less than three times the number of 18¢ stamps. The total value of all the stamps is $1.29. How many 15¢ stamps are in the drawer?

19) The total value of dimes and quarters in a bank is $6.05. There are six more quarters than dimes. Find the number of each type of coin in the bank.

20) A child's piggy bank contains 44 coins in quarters and dimes. The coins have a total value of $8.60. Find the number of quarters in the bank.

21) A coin bank contains nickels and dimes. The number of dimes is 10 less than twice the number of nickels. The total value of all the coins is $2.75. Find the number of each type of coin in the bank.

22) A total of 26 bills are in a cash box. Some of the bills are one dollar bills, and the rest are five dollar bills. The total amount of cash in the box is $50. Find the number of each type of bill in the cash box.

23) A bank teller cashed a check for $200 using twenty dollar bills and ten dollar bills. In all, twelve bills were handed to the customer. Find the number of twenty dollar bills and the number of ten dollar bills.

24) A collection of stamps consists of 22¢ stamps and 40¢ stamps. The number of 22¢ stamps is three more than four times the number of 40¢ stamps. The total value of the stamps is $8.34. Find the number of 22¢ stamps in the collection.

25) A total of $27000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is $3385. How much was invested at each rate?

26) A total of $50000 is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is $3250. How much was invested at each rate?

27) A total of $9000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is $1030. How much was invested at each rate?

28) A total of $18000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is $1248. How much was invested at each rate?

29) An inheritance of $10000 is invested in 2 ways, part at 9.5% and the remainder at 11%. The combined annual interest was $1038.50. How much was invested at each rate?

30) Kerry earned a total of $900 last year on his investments. If $7000 was invested at a certain rate of return and $9000 was invested in a fund with a rate that was 2% higher, find the two rates of interest.

31) Jason earned $256 interest last year on his investments. If $1600 was invested at a certain rate of return and $2400 was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

32) Millicent earned $435 last year in interest. If $3000 was invested at a certain rate of return and $4500 was invested in a fund with a rate that was 2% lower, find the two rates of interest.

33) A total of $8500 is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is $385. How much was invested at each rate?

34) A total of $12000 was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is $1005. How much was invested at each rate?

35) A total of $15000 is invested, part of it at 8% and the rest at 11%. The total interest after one year is $1455. How much was invested at each rate?

36) A total of $17500 is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is $1227.50. How much was invested at each rate?

37) A total of $6000 is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is $300. How much was invested at each rate?

38) A total of $14000 is invested, part of it at 5.5% and the rest at 9%. The total interest after one year is $910. How much was invested at each rate?

39) A total of $11000 is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is $797. How much was invested at each rate?

40) An investment portfolio earned $2010 in interest last year. If $3000 was invested at a certain rate of return and $24000 was invested in a fund with a rate that was 4% lower, find the two rates of interest.

41) Samantha earned $1480 in interest last year on her investments. If $5000 was invested at a certain rate of return and $11000 was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

2.4 Answers

1) 33 quarters, 70 dimes
2) 26 half dollars, 8 nickels
3) 236 adult, 342 child
4) 9 dimes, 12 quarters
5) 9 nickels, 18 dimes
6) 7 quarters, 4 half dollars
7) 9 nickels, 18 dimes
8) 25 nickels, 20 dimes
9) 203 adults, 226 child
10) 130 adults, 70 students
11) 128 card, 75 non-card
12) 73 hot dogs, 58 hamburgers
13) 135 students, 97 non-students
14) 12 dimes, 15 quarters
15) 13 nickels, 5 dimes
16) 8 20¢ stamps, 32 25¢ stamps
17) 6 15¢ stamps, 9 25¢ stamps
18) 5 15¢ stamps
19) 13 dimes, 19 quarters
20) 28 quarters
21) 15 nickels, 20 dimes
22) 20 $1 bills, 6 $5 bills
23) 8 $20 bills, 4 $10 bills
24) 27 22¢ stamps
25) $12500 @ 12% , $14500 @ 13%
26) $20000 @ 5%, $30000 @ 7.5%
27) $2500 @ 10%, $6500 @ 12%
28) $12400 @ 6%, $5600 @ 9%
29) $4100 @ 9.5%, $5900 @ 11%
30) $7000 @ 4.5%, $9000 @ 6.5%
31) $1600 @ 4%; $2400 @ 8%
32) $3000 @ 7%, $4500 @ 5%
33) $3500 @ 6%; $5000 @ 3.5%
34) $7000 @ 9%, $5000 @ 7.5%
35) $6500 @ 8%; $8500 @ 11%
36) $12000 @ 7.25%, $5500 @ 6.5%
37) $3000 @ 4.25%; $3000 @ 5.75%
38) $10000 @ 5.5%, $4000 @ 9%
39) $7500 @ 6.8%; $3500 @ 8.2%
40) $3000 @ 11%; $24000 @ 7%
41) $5000 @ 12%, $11000 @ 8%

Chapter 3: Polynomials

Section 3.1: Exponent Properties

Objective: Simplify expressions using the properties of exponents.

Problems with exponents can often be simplified using a few basic exponent properties. Exponents represent repeated multiplication. We will use this fact to discover the important properties.

World View Note: The word exponent comes from the Latin “expo” meaning “out of” and “ponere” meaning “place”. While there is some debate, it seems that the Babylonians living in Iraq were the first to do work with exponents (dating back to the 23rd century BC or earlier).

Example 1. Simplify.

a^3a^2	Expand exponents to multiplication problem
$(aaa)(aa)$	Now we have $5a$ ’*s* being multiplied together
a^5	Our Solution

A quicker method to arrive at our answer would have been to just add the exponents: $a^3a^2 = a^{3+2} = a^5$ This is known as the **product rule of exponents**.

Product Rule of Exponents: $\boldsymbol{a^m a^n = a^{m+n}}$

The product rule of exponents can be used to simplify many problems. We will add the exponents on like bases. This is shown in the following examples.

Example 2. Simplify.

$3^2 \cdot 3^6 \cdot 3$	Same base; add exponents $2+6+1$
3^9	Our Solution

Example 3. Simplify.

$2x^3y^5z \cdot 5xy^2z^3$	Multiply $2 \cdot 5$; add exponents on x, y and z
$10x^4y^7z^4$	Our Solution

Rather than multiplying, we will now try to divide with exponents.

Example 4. Simplify.

$\dfrac{a^5}{a^2}$	Expand exponents
$\dfrac{aaaaa}{aa}$	Divide out two of the a ’s

$$aaa \quad \text{Convert to exponents}$$
$$a^3 \quad \text{Our Solution}$$

A quicker method to arrive at the solution would have been to just subtract the exponents: $\frac{a^5}{a^2} = a^{5-2} = a^3$. This is known as the quotient rule of exponents.

Quotient Rule of Exponents: $\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$

The quotient rule of exponents can similarly be used to simplify exponent problems by subtracting exponents on like bases. This is shown in the following examples.

Example 5. Simplify.

$$\frac{7^{13}}{7^5} \quad \text{Same base; subtract exponents}$$
$$7^8 \quad \text{Our Solution}$$

Example 6. Simplify.

$$\frac{5a^3b^5c^2}{2ab^3c} \quad \text{Subtract exponents } a,\ b \text{ and } c$$
$$\frac{5}{2}a^2b^2c \quad \text{Our Solution}$$

A third property we will look at will have an exponent expression raised to a second exponent. This is investigated in the following example.

Example 7. Simplify.

$$\left(a^2\right)^3 \quad \text{Notice } a^2 \text{ three times}$$
$$a^2 \cdot a^2 \cdot a^2 \quad \text{Add exponents}$$
$$a^6 \quad \text{Our solution}$$

A quicker method to arrive at the solution would have been to just multiply the exponents: $\left(a^2\right)^3 = a^{2\cdot 3} = a^6$. This is known as the power of a power rule of exponents.

Power of a Power Rule of Exponents: $\boldsymbol{\left(a^m\right)^n = a^{mn}}$

This property is often combined with two other properties which we will investigate now.

Example 8. Simplify.

$(ab)^3$	Notice (ab) three times
$(ab)(ab)(ab)$	Three a's and three b's can be written with exponents
a^3b^3	Our solution

A quicker method to arrive at the solution would have been to take the exponent of three and put it on each factor in parentheses: $(ab)^3 = a^3b^3$. This is known as the power of a product rule of exponents.

Power of a Product Rule of Exponents: $\boldsymbol{(ab)^m = a^mb^m}$

It is important to be careful to only use the power of a product rule with multiplication inside parentheses. This property does NOT work if there is addition or subtraction.

Warning!

$(a+b)^m \neq a^m + b^m$ These are **NOT** equal; beware of this error!

Another property that is very similar to the power of a product rule is considered next.

Example 9. Simplify.

$\left(\frac{a}{b}\right)^3$	Notice the fraction three times
$\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)$	Multiply fractions across the top and bottom and use exponents
$\frac{a^3}{b^3}$	Our solution

A quicker method to arrive at the solution would have been to put the exponent on every factor in both the numerator and denominator: $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$. This is known as the power of a quotient rule of exponents.

Power of a Quotient Rule of Exponents: $\boldsymbol{\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}}$

The power of a power, product and quotient rules are often used together to simplify expressions. This is shown in the following examples.

Example 10. Simplify.

$$(x^3yz^2)^4 \quad \text{Put exponent of 4 on each factor; multiply powers}$$

$$x^{12}y^4z^8 \quad \text{Our Solution}$$

Example 11. Simplify.

$$\left(\frac{a^3b}{c^8d^5}\right)^2 \quad \text{Put exponent of 2 on each factor; multiply powers}$$

$$\frac{a^6b^2}{c^{16}d^{10}} \quad \text{Our Solution}$$

As we multiply exponents, it is important to remember these properties apply to exponents and not the bases. An expression such as 5^3 does not mean we multiply 5 by 3; instead we multiply 5 three times, $5\times5\times5=125$. This is shown in the next example.

Example 12. Simplify.

$$(4x^2y^5)^3 \quad \text{Put exponent of 3 on each factor; multiply powers}$$

$$4^3x^6y^{15} \quad \text{Evaluate } 4^3$$

$$64x^6y^{15} \quad \text{Our Solution}$$

In the previous example we did not put the 3 on the 4 and multiply to get 12. This would have been incorrect. Never multiply a base by the exponent. These properties pertain to exponents only and not the bases.

In this lesson we have discussed 5 different exponent properties. These rules are summarized in the following table.

Rules of Exponents

Product Rule of Exponents	$a^ma^n = a^{m+n}$
Quotient Rule of Exponents	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power Rule of Exponents	$(a^m)^n = a^{mn}$
Power of a Product Rule of Exponents	$(ab)^m = a^mb^m$
Power of a Quotient Rule of Exponents	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

These five properties are often combined in the same problem. Often there is a bit of flexibility as to which property is used first. However, order of operations still applies to a problem. For this reason it is the suggestion of the author to simplify inside parentheses first;

then simplify any exponents (using power rules); and finally simplify any multiplication or division (using product and quotient rules). This is illustrated in the next few examples.

Example 13. Simplify.

$(4x^3y\cdot 5x^4y^2)^3$	In parentheses simplify using product rule; add exponents
$(20x^7y^3)^3$	In parentheses simplify using product rule; add exponents
$20^3x^{21}y^9$	Evaluate 20^3
$8000x^{21}y^9$	Our Solution

Example 14. Simplify.

$7a^3(2a^4)^3$	Parentheses are already simplified; use power rule
$7a^3(8a^{12})$	Use product rule; add exponents; multiply numbers
$56a^{15}$	Our Solution

Example 15. Simplify.

$\dfrac{3a^3b\cdot 10a^4b^3}{2a^4b^2}$	Simplify numerator with product rule; add exponents
$\dfrac{30a^7b^4}{2a^4b^2}$	Now use the quotient rule to subtract exponents
$15a^3b^2$	Our Solution

Example 16. Simplify.

$\dfrac{3m^8n^{12}}{(m^2n^3)^3}$	Use power rule in denominator
$\dfrac{3m^8n^{12}}{m^6n^9}$	Use quotient rule
$3m^2n^3$	Our Solution

Example 17. Simplify.

$\left(\dfrac{3ab^2(2a^4b^2)^3}{6a^5b^7}\right)^2$	Simplify inside parentheses first; use power rule in numerator

$$\left(\frac{3ab^2(8a^{12}b^6)}{6a^5b^7}\right)^2 \quad \text{Simplify numerator; use product rule}$$

$$\left(\frac{24a^{13}b^8}{6a^5b^7}\right)^2 \quad \text{Simplify; use quotient rule}$$

$$(4a^8b)^2 \quad \text{Parentheses are simplified; use power rule}$$

$$16a^{16}b^2 \quad \text{Our Solution}$$

Clearly these problems can quickly become quite involved. Remember to follow order of operations as a guide; simplify inside parentheses first; then use power rules followed by the product and quotient rules.

3.1 Practice

Simplify each expression.

1) $4 \cdot 4^4 \cdot 4^4$
2) $4 \cdot 4^4 \cdot 4^2$
3) $4 \cdot 2^2$
4) $3 \cdot 3^3 \cdot 3^2$
5) $3m \cdot 4mn$
6) $3x \cdot 4x^2$
7) $2m^4n^2 \cdot 4nm^2$
8) $x^2y^4 \cdot xy^2$
9) $(3^3)^4$
10) $(4^3)^4$
11) $(4^4)^2$
12) $(3^2)^3$
13) $(2u^3v^2)^2$
14) $(xy)^3$
15) $(2a^4)^4$
16) $(2xy)^4$
17) $\dfrac{4^5}{4^3}$
18) $\dfrac{3^7}{3^3}$
19) $\dfrac{3^2}{3}$
20) $\dfrac{3^4}{3}$
21) $\dfrac{3nm^2}{3n}$
22) $\dfrac{x^2y^4}{4xy}$
23) $\dfrac{4x^3y^4}{3xy^3}$
24) $\dfrac{xy^3}{4xy}$

25) $\left(x^3y^4 \cdot 2x^2y^3\right)^2$

26) $\left(u^2v^2 \cdot 2u^4\right)^3$

27) $2x\left(x^4y^4\right)^4$

28) $\dfrac{3vu^5 \cdot 2v^3}{uv^2 \cdot 2u^3v}$

29) $\dfrac{2x^7y^5}{3x^3y \cdot 4x^2y^3}$

30) $\dfrac{2ba^7 \cdot 2b^4}{ba^2 \cdot 3a^3b^4}$

31) $\left(\dfrac{(2x)^3}{x^3}\right)^2$

32) $\dfrac{2a^2b^2a^7}{(ba^4)^2}$

33) $\left(\dfrac{2y^{17}}{(2x^2y^4)^4}\right)^3$

34) $\dfrac{yx^2 \cdot (y^4)^2}{2y^4}$

35) $\left(\dfrac{2mn^4 \cdot 2m^4n^4}{mn^4}\right)^3$

36) $\dfrac{n^3(n^4)^2}{2mn}$

37) $\dfrac{2xy^5 \cdot 2x^2y^3}{2xy^4 \cdot y^3}$

38) $\dfrac{(2y^3x^2)^2}{2x^2y^4 \cdot x^2}$

39) $\dfrac{q^3r^2 \cdot (2p^2q^2r^3)^2}{2p^3}$

40) $\dfrac{2x^4y^5 \cdot 2z^{10}x^2y^7}{(xy^2z^2)^4}$

41) $\left(\dfrac{zy^3 \cdot z^3x^4y^4}{x^3y^3z^3}\right)^4$

42) $\left(\dfrac{2q^3p^3r^4 \cdot 2p^3}{(qrp^3)^2}\right)^4$

3.1 Answers

1) $4^9 = 262{,}144$
2) $4^7 = 16{,}384$
3) $2^4 = 16$
4) $3^6 = 729$
5) $12m^2n$
6) $12x^3$
7) $8m^6n^3$
8) x^3y^6
9) $3^{12} = 531{,}441$
10) $4^{12} = 16{,}777{,}216$
11) $4^8 = 65{,}536$
12) $3^6 = 729$
13) $4u^6v^4$
14) x^3y^3
15) $16a^{16}$
16) $16x^4y^4$
17) $4^2 = 16$
18) $3^4 = 81$
19) 3
20) $3^3 = 27$
21) m^2
22) $\dfrac{xy^3}{4}$
23) $\dfrac{4x^2y}{3}$
24) $\dfrac{y^2}{4}$
25) $4x^{10}y^{14}$
26) $8u^{18}v^6$
27) $2x^{17}y^{16}$
28) $3uv$
29) $\dfrac{x^2y}{6}$

30) $\frac{4a^2}{3}$

31) 64

32) $2a$

33) $\frac{y^3}{512x^{24}}$

34) $\frac{y^5x^2}{2}$

35) $64m^{12}n^{12}$

36) $\frac{n^{10}}{2m}$

37) $2x^2y$

38) $2y^2$

39) $2q^7r^8p$

40) $4x^2y^4z^2$

41) $x^4y^{16}z^4$

42) $256q^4r^8$

Section 3.2: Negative Exponents

Objective: Simplify expressions with negative exponents using the properties of exponents.

There are a few special exponent properties that deal with exponents that are not positive. The first is considered in the following example, which is written out in two different ways.

Example 1. Simplify.

$\dfrac{a^3}{a^3}$	Use quotient rule; subtract exponents
a^0	Our Solution; now consider the problem in the second way
$\dfrac{a^3}{a^3}$	Rewrite exponents; use repeated multiplication
$\dfrac{aaa}{aaa}$	Reduce out all the a's
$\dfrac{1}{1}=1$	Our Solution; combine the two solutions, get:
$a^0=1$	Our Final Solution

This final result is an important property known as the zero power rule of exponents.

Zero Power Rule of Exponents: $a^0=1$

Any non-zero number or expression raised to the zero power will always be 1. This is illustrated in the following example.

Example 2. Simplify.

$(3x^2)^0$	Zero power rule
1	Our Solution

Here we are assuming that x is not 0. If $x=0$, then $(3\cdot 0^2)^0$ would equal 0, not 1. Another property we will consider here deals with negative exponents. Again we will solve the following example in two ways.

Example 3. Simplify and write the answer using only positive exponents.

$\dfrac{a^3}{a^5}$	Use quotient rule; subtract exponents
a^{-2}	Our Solution; solve this problem another way

$\frac{a^3}{a^5}$ Rewrite exponents; use repeated multiplication

$\frac{aaa}{aaaaa}$ Reduce three a's out of top and bottom

$\frac{1}{aa}$ Simplify to exponents

$\frac{1}{a^2}$ Our Solution; combine the two solutions; get:

$a^{-2} = \frac{1}{a^2}$ Our Final Solution

This example illustrates an important property of exponents. Negative exponents yield the reciprocal of the base. Once we take the reciprocal, the exponent is now positive. Also, it is important to note that a negative exponent does not mean that the expression is negative; only that we need the reciprocal of the base. The rules of negative exponents follow.

Rules of Negative Exponents:

$$\boldsymbol{a^{-m} = \frac{1}{a^m}}$$

$$\boldsymbol{\frac{1}{a^{-m}} = a^m}$$

$$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$$

Negative exponents can be combined in several different ways. As a general rule, if we think of our expression as a fraction, negative exponents in the numerator must be moved to the denominator; likewise, negative exponents in the denominator need to be moved to the numerator. When the base with exponent moves, the exponent is now positive. This is illustrated in the following example.

Example 4. Simplify and write the answer using only positive exponents.

$\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2}$ Move negative exponents on b, d, and e; Exponents become positive

$\frac{a^3cde^4}{2b^2f^2}$ Our Solution

As we simplified our fraction, we took special care to move the bases that had a negative exponent, but the expression itself did not become negative because of those exponents. Also, it is important to remember that exponents only effect the base they are attached to. The 2 in the denominator of the above example does not have an exponent on it, so it does not move with the d.

We now have the following nine properties of exponents. It is important that we are very familiar with all of them. Note that when simplifying expressions involving exponents, the final form is usually written using only positive exponents.

Properties of Exponents

$\boldsymbol{a^m a^n = a^{m+n}}$	$\boldsymbol{(ab)^m = a^m b^m}$	$\boldsymbol{a^{-m} = \frac{1}{a^m}}$
$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$	$\boldsymbol{\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}}$	$\boldsymbol{\frac{1}{a^{-m}} = a^m}$
$\boldsymbol{\left(a^m\right)^n = a^{mn}}$	$\boldsymbol{a^0 = 1}$	$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$

World View Note: Nicolas Chuquet, the French mathematician of the 15th century, wrote $12^{1\bar{m}}$ to indicate $12x^{-1}$. This was the first known use of the negative exponent.

Simplifying with negative exponents is much the same as simplifying with positive exponents. It is the advice of the author to keep the negative exponents until the end of the problem and then move them around to their correct location (numerator or denominator). As we do this, it is important to be very careful of rules for adding, subtracting, and multiplying with negatives. This is illustrated in the following examples.

Example 5. Simplify and write the answer using only positive exponents.

$\dfrac{4x^{-5}y^{-3}\cdot 3x^3y^{-2}}{6x^{-5}y^3}$	Simplify numerator with product rule; add exponents
$\dfrac{12x^{-2}y^{-5}}{6x^{-5}y^3}$	Use quotient rule; subtract exponents; be careful with negatives $(-2)-(-5)=(-2)+5=3$
	$(-5)-3=(-5)+(-3)=-8$
$2x^3y^{-8}$	Move negative exponent to denominator; exponent becomes positive
$\dfrac{2x^3}{y^8}$	Our Solution

Example 6. Simplify and write the answer using only positive exponents.

$\dfrac{(3ab^3)^{-2}ab^{-3}}{2a^{-4}b^0}$	In numerator, use power rule with -2; multiply exponents In denominator, $b^0=1$
$\dfrac{3^{-2}a^{-2}b^{-6}ab^{-3}}{2a^{-4}}$	In numerator, use product rule; add exponents
$\dfrac{3^{-2}a^{-1}b^{-9}}{2a^{-4}}$	Use quotient rule; subtract exponents; be careful with negatives $(-1)-(-4)=(-1)+4=3$

$\dfrac{3^{-2}a^3b^{-9}}{2}$ Move 3 and b to denominator because of negative exponents; Exponents become positive

$\dfrac{a^3}{3^2 2b^9}$ Evaluate $3^2 * 2 = 9 * 2 = 18$

$\dfrac{a^3}{18b^9}$ Our Solution

In the previous example it is important to point out that when we simplified 3^{-2}, we moved the three to the denominator and the exponent became positive. We did not make the number negative! Negative exponents never make the bases negative; they simply mean we have to take the reciprocal of the base. One final example with negative exponents is given here.

Example 7. Simplify and write the answer using only positive exponents.

$\left(\dfrac{3x^{-2}y^5z^3 \cdot 6x^{-6}y^{-2}z^{-3}}{9(x^2y^{-2})^{-3}}\right)^{-3}$ In numerator, use product rule; adding exponents
In denominator, use power rule, multiplying exponents

$\left(\dfrac{18x^{-8}y^3z^0}{9x^{-6}y^6}\right)^{-3}$ Use quotient rule to subtract exponents, be careful with negatives:
$(-8)-(-6)=(-8)+6=-2$
$3-6=3+(-6)=-3$

$(2x^{-2}y^{-3}z^0)^{-3}$ Parentheses are done; use power rule with -3

$2^{-3}x^6y^9z^0$ Move 2 with negative exponent down and $z^0 = 1$; exponent of 2 becomes positive

$\dfrac{x^6y^9}{2^3}$ Evaluate 2^3

$\dfrac{x^6y^9}{8}$ Our Solution

3.2 Practice

Simplify each expression. Your answer should contain only positive exponents.

1) $2x^4y^{-2}\cdot(2xy^3)^4$
2) $2a^{-2}b^{-3}\cdot(2a^0b^4)^4$
3) $(a^4b^{-3})^3\cdot 2a^3b^{-2}$
4) $2x^3y^2\cdot(2x^3)^0$
5) $(2x^2y^2)^4\cdot x^{-4}$
6) $(m^0n^3\cdot 2m^{-3}n^{-3})^0$
7) $(x^3y^4)^3\cdot x^{-4}y^4$
8) $2m^{-1}n^{-3}\cdot(2m^{-1}n^{-3})^4$
9) $\dfrac{2x^{-3}y^2}{3x^{-3}y^3\cdot 3x^0}$
10) $\dfrac{3y^3}{3yx^3\cdot 2x^4y^{-3}}$
11) $\dfrac{4xy^{-3}\cdot x^{-4}y^0}{4y^{-1}}$
12) $\dfrac{3x^3y^2}{4y^{-2}\cdot 3x^{-2}y^{-4}}$
13) $\dfrac{u^2v^{-1}}{2u^0v^4\cdot 2uv}$
14) $\dfrac{2xy^2\cdot 4x^3y^{-4}}{4x^{-4}y^{-4}\cdot 4x}$
15) $\dfrac{u^2}{4u^0v^3\cdot 3v^2}$
16) $\dfrac{2x^{-2}y^2}{4yx^2}$
17) $\dfrac{2y}{(x^0y^2)^4}$
18) $\dfrac{(a^4)^4}{2b}$
19) $\left(\dfrac{2a^2b^3}{a^{-1}}\right)^4$

20) $\left(\frac{2y^{-4}}{x^2}\right)^{-2}$

21) $\frac{2nm^4}{(2m^2n^2)^4}$

22) $\frac{2y^2}{(x^4y^0)^{-4}}$

23) $\frac{(2mn)^4}{m^0n^{-2}}$

24) $\frac{2x^{-3}}{(x^4y^{-3})^{-1}}$

25) $\frac{y^3 \cdot x^{-3}y^2}{(x^4y^2)^3}$

26) $\frac{2x^{-2}y^0 \cdot 2xy^4}{(xy^0)^{-1}}$

27) $\frac{2u^{-2}v^3 \cdot (2uv^4)^{-1}}{2u^{-4}v^0}$

28) $\frac{2yx^2 \cdot x^{-2}}{(2x^0y^4)^{-1}}$

29) $\left(\frac{2x^0 \cdot y^4}{y^4}\right)^3$

30) $\frac{u^{-3}v^{-4}}{2v(2u^{-3}v^4)^0}$

31) $\frac{y(2x^4y^2)^2}{2x^4y^0}$

32) $\frac{b^{-1}}{(2a^4b^0)^0 \cdot 2a^{-3}b^2}$

33) $\frac{2yzx^2}{2x^4y^4z^{-2} \cdot (zy^2)^4}$

34) $\frac{2b^4c^{-2} \cdot (2b^3c^2)^{-4}}{a^{-2}b^4}$

35) $\frac{2kh^0 \cdot 2h^{-3}k^0}{(2kj^3)^2}$

36) $\left(\frac{(2x^{-3}y^0z^{-1})^3 \cdot x^{-3}y^2}{2x^3}\right)^{-2}$

37) $\dfrac{(cb^3)^2 \cdot 2a^{-3}b^2}{(a^3b^{-2}c^3)^3}$

38) $\dfrac{2q^4 \cdot m^2p^2q^4}{(2m^{-4}p^2)^3}$

39) $\dfrac{(yx^{-4}z^2)^{-1}}{z^3 \cdot x^2y^3z^{-1}}$

40) $\dfrac{2mpn^{-3}}{(m^0n^{-4}p^2)^3 \cdot 2n^2p^0}$

3.2 Answers

1) $32x^8y^{10}$
2) $\frac{32b^{13}}{a^2}$
3) $\frac{2a^{15}}{b^{11}}$
4) $2x^3y^2$
5) $16x^4y^8$
6) 1
7) $y^{16}x^5$
8) $\frac{32}{m^5n^{15}}$
9) $\frac{2}{9y}$
10) $\frac{y^5}{2x^7}$
11) $\frac{1}{y^2x^3}$
12) $\frac{y^8x^5}{4}$
13) $\frac{u}{4v^6}$
14) $\frac{x^7y^2}{2}$
15) $\frac{u^2}{12v^5}$
16) $\frac{y}{2x^4}$
17) $\frac{2}{y^7}$
18) $\frac{a^{16}}{2b}$
19) $16a^{12}b^{12}$
20) $\frac{y^8x^4}{4}$

21) $\frac{1}{8m^4n^7}$

22) $2x^{16}y^2$

23) $16n^6m^4$

24) $\frac{2x}{y^3}$

25) $\frac{1}{x^{15}y}$

26) $4y^4$

27) $\frac{u}{2v}$

28) $4y^5$

29) 8

30) $\frac{1}{2u^3v^5}$

31) $2y^5x^4$

32) $\frac{a^3}{2b^3}$

33) $\frac{1}{x^2y^{11}z}$

34) $\frac{a^2}{8c^{10}b^{12}}$

35) $\frac{1}{h^3kj^6}$

36) $\frac{x^{30}z^6}{16y^4}$

37) $\frac{2b^{14}}{a^{12}c^7}$

38) $\frac{m^{14}q^8}{4p^4}$

39) $\frac{x^2}{y^4z^4}$

40) $\frac{mn^7}{p^5}$

Section 3.3: Scientific Notation

Objective: Multiply and divide expressions using scientific notation and exponent properties.

One application of exponent properties comes from scientific notation. Scientific notation is used to represent very large or very small numbers. An example of really large numbers would be the distance that light travels in a year in miles.
An example of really small numbers would be the mass of a single hydrogen atom in grams. Doing basic operations, such as multiplication and division with these numbers, would normally be very cumbersome. However, our exponent properties make this process much simpler.
First, we will take a look at what scientific notation is. Scientific notation has two parts: the **mantissa**, a, which is a number between one and ten (it can be equal to one, but not ten), and that number multiplied by ten to some exponent, n, where n is an integer.

Scientific Notation: $\pm a \times 10^n$ **where** $1 \le a < 10$

The exponent, n, is very important to how we convert between scientific notation and normal numbers, or standard notation. The exponent tells us how many times we will multiply by 10. Multiplying by 10 moves the decimal point one place to the right. So the exponent will tell us how many times the decimal point moves between scientific notation and standard notation. To decide which direction to move the decimal (left or right), we simply need to remember that positive exponents mean we have a big number (bigger than ten) in standard notation, and negative exponents mean we have a small number (less than one) in standard notation.
Keeping this in mind, we can easily make conversions between standard notation and scientific notation.

Example 1. Convert to scientific notation.

14,200	Put decimal after first nonzero number
1.42	Exponent is how many places decimal moved; 4 places
$\times 10^4$	Positive exponent; standard notation is big
1.42×10^4	Our Solution

Example 2. Convert to scientific notation.

0.0042	Put decimal after first nonzero number
4.2	Exponent is how many places decimal moved, 3 places
$\times 10^{-3}$	Negative exponent; standard notation is small
4.2×10^{-3}	Our Solution

Example 3. Convert to standard notation.

3.21×10^{5}	Positive exponent means big number, Move decimal right 5 places
$321{,}000$	Our Solution

Example 4. Convert to standard notation.

7.4×10^{-3}	Negative exponent means small number Move decimal left 3 places
0.0074	Our Solution

Converting between standard notation and scientific notation is important in understanding how scientific notation works and what it does. Here, our main interest is to be able to multiply and divide numbers in scientific notation using exponent properties. First, do the operation with the front numbers (multiply or divide); then use exponent properties to simplify the 10's. Scientific notation is the only time where we will be allowed to have negative exponents in our final solution. The negative exponent simply informs us that we are dealing with small numbers. Consider the following examples.

Example 5. Simplify and write the answer in scientific notation.

$(2.1\times10^{-7})(3.7\times10^{5})$	Deal with numbers and 10's separately
$(2.1)(3.7)=7.77$	Multiply numbers
$10^{-7}10^{5}=10^{-2}$	Use product rule on 10's; add exponents
7.77×10^{-2}	Our Solution

Example 6. Simplify and write the answer in scientific notation.

$\dfrac{4.96\times10^{4}}{3.1\times10^{-3}}$	Deal with numbers and 10's separately
$\dfrac{4.96}{3.1}=1.6$	Divide numbers
$\dfrac{10^{4}}{10^{-3}}=10^{7}$	Use quotient rule; subtract exponents; be careful with negatives Be careful with negatives, $4-(-3)=4+3=7$
1.6×10^{7}	Our Solution

Example 7. Simplify and write the answer in scientific notation.

$(1.8\times10^{-4})^{3}$	Use power rule to deal with numbers and 10's separately
$1.8^{3}=5.832$	Evaluate 1.8^{3}
$(10^{-4})^{3}=10^{-12}$	Use power of a power rule; multiply exponents

5.832×10^{-12} Our Solution

Often when we multiply or divide in scientific notation, the end result is not in scientific notation. We will have to convert the front number or mantissa to scientific notation; then combine the 10's using the product property of exponents to add the exponents. This is shown in the following examples.

Example 8. Simplify and write the answer in scientific notation.

$(4.7\times10^{-3})(6.1\times10^{9})$	Deal with numbers and 10's separately
$(4.7)(6.1)=28.67$	Multiply numbers
2.867×10^{1}	Convert this number to scientific notation
$\mathbf{10}^{1}10^{-3}10^{9}=10^{7}$	Use product rule; add exponents; use 10^{1} from conversion
2.867×10^{7}	Our Solution

World View Note: Archimedes (287 BC - 212 BC), the Greek mathematician, developed a system for representing large numbers using a system very similar to scientific notation. He used his system to calculate the number of grains of sand it would take to fill the universe. His conclusion was 1063 grains of sand, because he figured the universe to have a diameter of 10^{14} stadia or about 2 light years.

Example 9. Simplify and write the answer in scientific notation.

$\dfrac{2.014\times10^{-3}}{3.8\times10^{-7}}$	Deal with numbers and 10's separately
$\dfrac{2.014}{3.8}=0.53$	Divide numbers
$0.53=5.3\times10^{-1}$	Change this number to scientific notation
$\dfrac{\mathbf{10}^{-1}\,10^{-3}}{10^{-7}}=10^{3}$	Use product and quotient rule; use 10^{-1} from the conversion Be careful with signs: $(-1)+(-3)-(-7)=(-1)+(-3)+7=3$
5.3×10^{3}	Our Solution

3.3 Practice

Write each number in scientific notation.

1) 885
2) 0.000744
3) 0.081
4) 1.09
5) 0.039
6) 15000

Write each number in standard notation.

7) 8.7×10^{5}
8) 2.56×10^{2}
9) 9×10^{-4}
10) 5×10^{4}
11) 2×10^{0}
12) 6×10^{-5}

Simplify. Write each answer in scientific notation and when necessary, round the mantissa to 3 decimal places.

13) $(7\times10^{-1})(2\times10^{-3})$
14) $(2\times10^{-6})(8.8\times10^{-5})$
15) $(5.26\times10^{-5})(3.16\times10^{-2})$
16) $(5.1\times10^{6})(9.84\times10^{-1})$
17) $(2.6\times10^{-2})(6\times10^{-2})$
18) $\dfrac{7.4\times10^{4}}{1.7\times10^{-4}}$
19) $\dfrac{4.9\times10^{1}}{2.7\times10^{-3}}$
20) $\dfrac{7.2\times10^{-1}}{7.32\times10^{-1}}$
21) $\dfrac{5.33\times10^{-6}}{9.62\times10^{-2}}$
22) $\dfrac{3.2\times10^{-3}}{5.02\times10^{0}}$
23) $(5.5\times10^{-5})^{2}$
24) $(9.6\times10^{3})^{-4}$

25) $(7.8\times10^{-2})^5$

26) $(5.4\times10^{6})^{-3}$

27) $(8.03\times10^{4})^{-4}$

28) $(6.88\times10^{-4})(4.23\times10^{1})$

29) $\dfrac{6.1\times10^{-6}}{5.1\times10^{-4}}$

30) $\dfrac{8.4\times10^{5}}{7\times10^{-2}}$

31) $(3.6\times10^{0})(6.1\times10^{-3})$

32) $(3.15\times10^{3})(8\times10^{-1})$

33) $(1.8\times10^{-5})^{-3}$

34) $\dfrac{9.58\times10^{-2}}{1.14\times10^{-3}}$

35) $\dfrac{9\times10^{4}}{7.83\times10^{-2}}$

36) $(8.3\times10^{1})^5$

37) $\dfrac{3.22\times10^{-3}}{7\times10^{-6}}$

38) $\dfrac{5\times10^{6}}{6.69\times10^{2}}$

3.3 Answers

1) 8.85×10^{2}
2) 7.44×10^{-4}
3) 8.1×10^{-2}
4) 1.09×10^{0}
5) 3.9×10^{-2}
6) 1.5×10^{4}
7) 870000
8) 256
9) 0.0009
10) 50000
11) 2
12) 0.00006
13) 1.4×10^{-3}
14) 1.76×10^{-10}
15) 1.662×10^{-6}
16) 5.018×10^{6}
17) 1.56×10^{-3}
18) 4.353×10^{8}
19) 1.815×10^{4}
20) 9.836×10^{-1}
21) 5.541×10^{-5}
22) 6.375×10^{-4}
23) 3.025×10^{-9}
24) 1.177×10^{-16}
25) 2.887×10^{-6}
26) 6.351×10^{-21}
27) 2.405×10^{-20}
28) 2.91×10^{-2}
29) 1.196×10^{-2}
30) 1.2×10^{7}
31) 2.196×10^{-2}
32) 2.52×10^{3}
33) 1.715×10^{14}
34) 8.404×10^{1}

35) 1.149×10^{6}

36) 3.939×10^{9}

37) 4.6×10^{2}

38) 7.474×10^{3}

39) 3.692×10^{-7}

Section 3.4: Introduction to Polynomials

Objective: Evaluate, add, and subtract polynomials.

Many applications in mathematics have to do with what are called polynomials. Polynomials are made up of terms. **Terms** are a product of numbers or variables. For example, $5x$, $2y^2$, -5, ab^3c and x are all terms. Terms are connected to each other by addition. Expressions are often named based on the number of terms in them. A **monomial** has one term, such as $3x^2$. A **binomial** has two terms, such as a^2-b^2. A **trinomial** has three terms, such as ax^2+bx+c. The term **polynomial** means many terms. Monomials, binomials, trinomials, and expressions with more terms all fall under the umbrella of "polynomials".

If we know what the variable in a polynomial represents, we can replace the variable with the number and evaluate the polynomial as shown in the following example.

Example 1. Evaluate the given expression.

$2x^2-4x+6$ when $x=-4$	Replace variable x with -4
$2(-4)^2-4(-4)+6$	Exponents first
$2(16)-4(-4)+6$	Multiplication (we can do all terms at one)
$32+16+6$	Add
54	Our Solution

It is important to be careful with negative variables and exponents. Remember the exponent only affects the base it is physically attached to. This means that $-3^2=-9$ because the exponent is only attached to the 3. Also, $(-3)^2=9$ because the exponent is attached to the parentheses and affects everything inside. When we replace a variable with parentheses like in the previous example, the substituted value is in parentheses. So the $(-4)^2=16$ in the example. However, consider the next example.

Example 2. Evaluate the given expression.

$-x^2+2x+6$ when $x=3$	Replace variable x with 3
$-(3)^2+2(3)+6$	Exponent only on the 3
$-9+2(3)+6$	Multiply
$-9+6+6$	Add
3	Our Solution

World View Note: Ada Lovelace in 1842 described a Difference Engine that would be used to calculate values of polynomials. Her work became the foundation for what would become the modern computer more than 100 years after her death from cancer. The programming language Ada was named in her honor.

Generally when working with polynomials, we do not know the value of the variable, so we will try and simplify instead. The simplest operation with polynomials is addition. When adding polynomials, we combine like terms. Consider the following example.

Example 3. Simplify.

$(4x^3 - 2x + 8) + (3x^3 - 9x^2 - 11)$	Remove the parentheses
$4x^3 - 2x + 8 + 3x^3 - 9x^2 - 11$	Combine like terms $4x^3 + 3x^3$ and $8 - 11$
$7x^3 - 9x^2 - 2x - 3$	Our Solution

Generally, final answers for polynomials are written so the exponent on the variable counts down. Example 3 demonstrates this with the exponent counting down 3, 2, 1, 0 (recall $x^0 = 1$). Subtracting polynomials is almost as fast. One extra step comes from the minus in front of the parentheses. When we have a negative in front of parentheses, we distribute it through, changing the signs of everything inside.

Example 4. Simplify.

$(5x^2 - 2x + 7) - (3x^2 + 6x - 4)$	Distribute negative through second part
$5x^2 - 2x + 7 - 3x^2 - 6x + 4$	Combine like terms: $5x^2 - 3x^2$, $-2x - 6x$ and $7 + 4$
$2x^2 - 8x + 11$	Our Solution

Addition and subtraction can also be combined into the same problem as shown in this final example.

Example 5. Simplify.

$(2x^2 - 4x + 3) + (5x^2 - 6x + 1) - (x^2 - 9x + 8)$	Distribute negative through
$2x^2 - 4x + 3 + 5x^2 - 6x + 1 - x^2 + 9x - 8$	Combine like terms
$6x^2 - x - 4$	Our Solution

3.4 Practice

Evaluate each expression in exercises 1-10.

1) $-a^3 - a^2 + 6a - 21$ when $a = -4$
2) $n^2 + 3n - 11$ when $n = -6$
3) $n^3 - 7n^2 + 15n - 20$ when $n = 2$
4) $n^3 - 9n^2 + 23n - 21$ when $n = 5$
5) $-5n^4 - 11n^3 - 9n^2 - n - 5$ when $n = -1$
6) $x^4 - 5x^3 - x + 13$ when $x = 5$
7) $x^2 + 9x + 23$ when $x = -3$
8) $-6x^3 + 41x^2 - 32x + 11$ when $x = 6$
9) $x^4 - 6x^3 + x^2 - 24$ when $x = 6$
10) $m^4 + 8m^3 + 14m^2 + 13m + 5$ when $m = -6$

Simplify each expression in exercises 11-42.

11) $(5p - 5p^4) - (8p - 8p^4)$
12) $(7m^2 + 5m^3) - (6m^3 - 5m^2)$
13) $(3n^2 + n^3) - (2n^3 - 7n^2)$
14) $(x^2 + 5x^3) + (7x^2 + 3x^3)$
15) $(8n + n^4) - (3n - 4n^4)$
16) $(3v^4 + 1) + (5 - v^4)$
17) $(1 + 5p^3) - (1 - 8p^3)$
18) $(6x^3 + 5x) - (8x + 6x^3)$
19) $(5n^4 + 6n^3) + (8 - 3n^3 - 5n^4)$
20) $(8x^2 + 1) - (6 - x^2 - x^4)$
21) $(3 + b^4) + (7 + 2b + b^4)$
22) $(1 + 6r^2) + (6r^2 - 2 - 3r^4)$
23) $(8x^3 + 1) - (5x^4 - 6x^3 + 2)$
24) $(4n^4 + 6) - (4n - 1 - n^4)$
25) $(2a + 2a^4) - (3a^2 - 5a^4 + 4a)$
26) $(6v + 8v^3) + (3 + 4v^3 - 3v)$
27) $(4p^2 - 3 - 2p) - (3p^2 - 6p + 3)$
28) $(7 + 4m + 8m^4) - (5m^4 + 1 + 6m)$
29) $(4b^3 + 7b^2 - 3) + (8 + 5b^2 + b^3)$

30) $(7n+1-8n^4)-(3n+7n^4+7)$

31) $(3+2n^2+4n^4)+(n^3-7n^2-4n^4)$

32) $(7x^2+2x^4+7x^3)+(6x^3-8x^4-7x^2)$

33) $(n-5n^4+7)+(n^2-7n^4-n)$

34) $(8x^2+2x^4+7x^3)+(7x^4-7x^3+2x^2)$

35) $(8r^4-5r^3+5r^2)+(2r^2+2r^3-7r^4+1)$

36) $(4x^3+x-7x^2)+(x^2-8+2x+6x^3)$

37) $(2n^2+7n^4-2)+(2+2n^3+4n^2+2n^4)$

38) $(7b^3-4b+4b^4)-(8b^3-4b^2+2b^4-8b)$

39) $(8-b+7b^3)-(3b^4+7b-8+7b^2)+(3-3b+6b^3)$

40) $(1-3n^4-8n^3)+(7n^4+2-6n^2+3n^3)+(4n^3+8n^4+7)$

41) $(8x^4+2x^3+2x)+(2x+2-2x^3-x^4)-(x^3+5x^4+8x)$

42) $(6x-5x^4-4x^2)-(2x-7x^2-4x^4-8)-(8-6x^2-4x^4)$

3.4 Answers

1) 3
2) 7
3) -10
4) -6
5) -7
6) 8
7) 5
8) -1
9) 12
10) -1
11) $3p^4-3p$
12) $-m^3+12m^2$
13) $-n^3+10n^2$
14) $8x^3+8x^2$
15) $5n^4+5n$
16) $2v^4+6$
17) $13p^3$
18) $-3x$
19) $3n^3+8$
20) x^4+9x^2-5
21) $2b^4+2b+10$
22) $-3r^4+12r^2-1$
23) $-5x^4+14x^3-1$
24) $5n^4-4n+7$
25) $7a^4-3a^2-2a$
26) $12v^3+3v+3$
27) p^2+4p-6
28) $3m^4-2m+6$
29) $5b^3+12b^2+5$
30) $-15n^4+4n-6$
31) n^3-5n^2+3
32) $-6x^4+13x^3$
33) $-12n^4+n^2+7$
34) $9x^4+10x^2$
35) $r^4-3r^3+7r^2+1$

36) $10x^3 - 6x^2 + 3x - 8$
37) $9n^4 + 2n^3 + 6n^2$
38) $2b^4 - b^3 + 4b^2 + 4b$
39) $-3b^4 + 13b^3 - 7b^2 - 11b + 19$
40) $12n^4 - n^3 - 6n^2 + 10$
41) $2x^4 - x^3 - 4x + 2$
42) $3x^4 + 9x^2 + 4x$

Section 3.5: Multiplying Polynomials

Objective: Multiply polynomials.

Multiplying polynomials can take several different forms based on what we are multiplying. We will first look at multiplying monomials; then we will multiply monomials by polynomials; and finish with multiplying polynomials by polynomials.

Multiplying monomials is done by multiplying the numbers or coefficients; then adding the exponents on like variable factors. This is shown in the next example.

Example 1. Simplify.

$(4x^3y^4z)(2x^2y^6z^3)$	Multiply numbers and add exponents for x, y, and z
$8x^5y^{10}z^4$	Our Solution

In the previous example, it is important to remember that the z has an exponent of 1 when no exponent is written. Thus, for our answer, the z has an exponent of $1+3=4$. Be very careful with exponents in polynomials. If we are adding or subtracting polynomials, the exponents will stay the same, but when we multiply (or divide) the exponents will be changing.

Next, we consider multiplying a monomial by a polynomial. We have seen this operation before when distributing through parentheses. Here we will see the exact same process.

Example 2. Simplify.

$4x^3(5x^2-2x+5)$	Distribute the $4x^3$; multiply numbers; add exponents
$20x^5-8x^4+20x^3$	Our Solution

Next, we have another example with more variables. When distributing, the exponents on a are added and the exponents on b are added.

Example 3. Simplify.

$2a^3b(3ab^2-4a)$	Distribute; multiply numbers; add exponents
$6a^4b^3-8a^4b$	Our Solution

There are several different methods for multiplying polynomials. Often students prefer the method they are first taught. Here, two methods will be discussed.

Both methods will be used to perform the same two multiplication problems.

Multiply by Distributing

Just as we distribute a monomial through parentheses, we can distribute an entire polynomial. As we do this, we take each term of the second polynomial and put it in front of the first polynomial.

Example 4. Simplify.

$(4x+7y)(3x-2y)$	Distribute $(4x+7y)$ through parentheses
$3x(\mathbf{4x+7y})-2y(\mathbf{4x+7y})$	Distribute the $3x$ and $-2y$
$12x^2+21xy-8xy-14y^2$	Combine like terms $21xy-8xy$
$12x^2+13xy-14y^2$	Our Solution

This example illustrates an important point that the negative/subtraction sign stays with the $2y$. On the second step, the negative is also distributed through the last set of parentheses.

Multiplying by distributing can easily be extended to problems with more terms. First, distribute the front parentheses onto each term; then distribute again.

Example 5. Simplify.

$(2x-5)(4x^2-7x+3)$	Distribute $(2x-5)$ through parentheses
$4x^2(\mathbf{2x-5})-7x(\mathbf{2x-5})+3(\mathbf{2x-5})$	Distribute again through each parentheses
$8x^3-20x^2-14x^2+35x+6x-15$	Combine like terms
$8x^3-34x^2+41x-15$	Our Solution

This process of multiplying by distributing can easily be reversed to do an important procedure known as factoring. Factoring will be addressed in a future lesson.

Multiply by FOIL

Another form of multiplying is known as FOIL. Using the FOIL method we multiply each term in the first binomial by each term in the second binomial. The letters of FOIL help us remember every combination. F stands for First, and we multiply the first term of each binomial. O stands for Outside, and we multiply the outside two terms. I stands for Inside, and we multiply the inside two terms. L stands for Last, and we multiply the last term of each binomial. This is shown in the next example.

Example 6. Simplify.

$(4x+7y)(3x-2y)$	Use FOIL to multiply
$(4x)(3x)=12x^2$	*F* – First terms $(4x)\,(3x)$
$(4x)(-2y)=-8xy$	*O* – Outside terms $(4x)\,(-2y)$
$(7y)(3x)=21xy$	*I* – Inside terms $(7y)\,(3x)$
$(7y)(-2y)=-14y^2$	*L* – Last terms $(7y)\,(-2y)$

$$12x^2 - 8xy + 21xy - 14y^2 \quad \text{Combine like terms } -8xy + 21xy$$
$$12x^2 + 13xy - 14y^2 \quad \text{Our Solution}$$

In reality, the FOIL method is a shortcut for distributing each of the terms in the first set of parentheses by all of the terms in the second set of parentheses. In the previous example, the first term, $4x$, is distributed through the $(3x-2y)$ and then the second term, $7y$, is distributed through the $(3x-2y)$. By distributing in this manner, it possible to multiply polynomials containing more than two terms.

Example 7. Simplify.

$(2x-5)(4x^2-7x+3)$	Distribute $2x$ and -5 to each term of the trinomial in the second set of parentheses
$(2x)(4x^2)+(2x)(-7x)+(2x)(3)-5(4x^2)-5(-7x)-5(3)$	Multiply out each term
$8x^3-14x^2+6x-20x^2+35x-15$	Combine like terms
$8x^3-34x^2+41x-15$	Our Solution

When we are multiplying a monomial by a polynomial by a polynomial, we can first multiply the polynomials; then distribute the monomial last. This is shown in the last example.

Example 8. Simplify.

$3(2x-4)(x+5)$	Multiply the binomials; use FOIL
$3(2x^2+10x-4x-20)$	Combine like terms
$3(2x^2+6x-20)$	Distribute the 3
$6x^2+18x-60$	Our Solution

A common error students do is distribute the three at the start into both parentheses. While we can distribute the 3 into the $(2x-4)$ factor, distributing into both would be wrong. Be careful of this error. This is why it is suggested to multiply the binomials first; then distribute the monomial last.

3.5 Practice

Find each product and simplify your answers.

1) $6(p-7)$
2) $4k(8k+4)$
3) $2(6x+3)$
4) $3n^2(6n+7)$
5) $5m^4(4m+4)$
6) $3(4r-7)$
7) $(4n+6)(8n+8)$
8) $(2x+1)(x-4)$
9) $(8b+3)(7b-5)$
10) $(r+8)(4r+8)$
11) $(4x+5)(2x+3)$
12) $(7n-6)(n+7)$
13) $(3v-4)(5v-2)$
14) $(6a+4)(a-8)$
15) $(6x-7)(4x+1)$
16) $(5x-6)(4x-1)$
17) $(5x+y)(6x-4y)$
18) $(2u+3v)(8u-7v)$
19) $(x+3y)(3x+4y)$
20) $(8u+6v)(5u-8v)$
21) $(7x+5y)(8x+3y)$
22) $(5a+8b)(a-3b)$
23) $(r-7)(6r^2-r+5)$
24) $(4x+8)(4x^2+3x+5)$
25) $(6n-4)(2n^2-2n+5)$
26) $(2b-3)(4b^2+4b+4)$
27) $(6x+3y)(6x^2-7xy+4y^2)$
28) $(3m-2n)(7m^2+6mn+4n^2)$
29) $(8n^2+4n+6)(6n^2-5n+6)$
30) $(2a^2+6a+3)(7a^2-6a+1)$
31) $(5k^2+3k+3)(3k^2+3k+6)$

32) $(7u^2+8uv-6v^2)(6u^2+4uv+3v^2)$

33) $3(3x-4)(2x+1)$

34) $5(x-4)(2x-3)$

35) $3(2x+1)(4x-5)$

36) $2(4x+1)(2x-6)$

37) $7(x-5)(x-2)$

38) $5(2x-1)(4x+1)$

39) $6(4x-1)(4x+1)$

40) $3(2x+3)(6x+9)$

3.5 Answers

1) $6p-42$
2) $32k^2+16k$
3) $12x+6$
4) $18n^3+21n^2$
5) $20m^5+20m^4$
6) $12r-21$
7) $32n^2+80n+48$
8) $2x^2-7x-4$
9) $56b^2-19b-15$
10) $4r^2+40r+64$
11) $8x^2+22x+15$
12) $7n^2+43n-42$
13) $15v^2-26v+8$
14) $6a^2-44a-32$
15) $24x^2-22x-7$
16) $20x^2-29x+6$
17) $30x^2-14xy-4y^2$
18) $16u^2+10uv-21v^2$
19) $3x^2+13xy+12y^2$
20) $40u^2-34uv-48v^2$
21) $56x^2+61xy+15y^2$
22) $5a^2-7ab-24b^2$
23) $6r^3-43r^2+12r-35$
24) $16x^3+44x^2+44x+40$
25) $12n^3-20n^2+38n-20$
26) $8b^3-4b^2-4b-12$
27) $36x^3-24x^2y+3xy^2+12y^3$
28) $21m^3+4m^2n-8n^3$
29) $48n^4-16n^3+64n^2-6n+36$
30) $14a^4+30a^3-13a^2-12a+3$
31) $15k^4+24k^3+48k^2+27k+18$
32) $42u^4+76u^3v+17u^2v^2-18v^4$
33) $18x^2-15x-12$
34) $10x^2-55x+60$

35) $24x^2 - 18x - 15$

36) $16x^2 - 44x - 12$

37) $7x^2 - 49x + 70$

38) $40x^2 - 10x - 5$

39) $96x^2 - 6$

40) $36x^2 + 108x + 81$

Section 3.6: Special Products

Objective: Recognize and use special product rules of a sum and a difference and perfect squares to multiply polynomials.

There are a few shortcuts that we can take when multiplying polynomials. If we can recognize them, the shortcuts can help us arrive at the solution much faster. These shortcuts will also be useful to us as our study of algebra continues.

The first shortcut is often called a **sum and a difference**. A sum and a difference is easily recognized as the numbers and variables are exactly the same, but the sign in the middle is different (one sum, one difference). To illustrate the shortcut, consider the following example, where we multiply using the distributing method.

Example 1. Simplify.

$(a+b)(a-b)$	Distribute $(a+b)$
$a(a+b)-b(a+b)$	Distribute a and $-b$
$a^2+ab-ab-b^2$	Combine like terms $ab-ab$
a^2-b^2	Our Solution

The important part of this example is that the middle terms subtracted to zero. Rather than going through all this work, when we have a sum and a difference, we will jump right to our solution by squaring the first term and squaring the last term, putting a subtraction between them. This is illustrated in the following example.

Example 2. Simplify.

$(x-5)(x+5)$	Recognize sum and difference
	Square both x and 5; put subtraction between the squares
x^2-25	Our Solution

This is much quicker than going through the work of multiplying and combining like terms. Often students ask if they can just multiply out using another method and not learn the shortcut. These shortcuts are going to be very useful when we get to factoring polynomials, or reversing the multiplication process. For this reason it is very important to be able to recognize these shortcuts. More examples are shown below.

Example 3. Simplify.

$(3x+7)(3x-7)$	Recognize sum and difference
	Square both $3x$ and 7; put subtraction between the squares
$9x^2-49$	Our Solution

Example 4. Simplify.

$(2x-6y)(2x+6y)$	Recognize sum and difference
	Square both $2x$ and $6y$; put subtraction between the squares
$4x^2-36y^2$	Our Solution

It is interesting to note that while we can multiply and get an answer like a^2-b^2 (with subtraction), it is impossible to multiply binomial expressions and end up with a product such as a^2+b^2 (with addition).

There is also a shortcut to multiply **a perfect square**, which is a binomial raised to the power two. The following example illustrates multiplying a perfect square.

Example 5. Simplify.

$(a+b)^2$	Square or multiply $(a+b)$ by itself
$(a+b)(a+b)$	Distribute $(a+b)$
$a(a+b)+b(a+b)$	Distribute again through final parentheses
$a^2+ab+ab+b^2$	Combine like terms $ab+ab$
$a^2+2ab+b^2$	Our Solution

This problem also helps us find our shortcut for multiplying. The first term in the answer is the **square of the first term** in the problem. The middle term is **2 times the first term times the second term**. The last term is the **square of the last term**. This can be shortened to square the first, twice the product, and square the last. If we can remember this shortcut, we can square any binomial. This is illustrated in the following example.

Example 6. Simplify.

$(x-5)^2$	Recognize perfect square
x^2	Square the first
$2(x)(-5)=-10x$	Twice the product
$(-5)^2=25$	Square the last
$x^2-10x+25$	Our Solution

Be very careful when squaring a binomial to **NOT** distribute the square through the parentheses. A common error is to do the following: $(x-5)^2=x^2-25$. Notice that both of these are missing the middle term, $-10x$. This is why it is important to use the shortcut to help us find the correct solution. Another important observation is that the second term in the solution always has the same sign as the second term in the problem. This is illustrated in the next examples.

Example 7. Simplify.

$(2x+5)^2$	Recognize perfect square
$(2x)^2 = 4x^2$	Square the first
$2(2x)(5) = 20x$	Twice the product
$5^2 = 25$	Square the last
$4x^2 + 20x + 25$	Our Solution

Example 8. Simplify.

$(3x-7y)^2$	Recognize perfect square
	Square the first; twice the product; square the last
$9x^2 - 42xy + 49y^2$	Our Solution

Example 9. Simplify.

$(5a+9b)^2$	Recognize perfect square
	Square the first; twice the product; square the last
$25a^2 + 90ab + 81b^2$	Our Solution

These two formulas will be important to commit to memory. The more familiar we are with them, the easier factoring, or multiplying in reverse, will be. The final example covers both types of problems (sum and difference and two perfect squares - one positive, one negative). Be sure to notice the difference between the examples and how each formula is used.

Example 10. Simplify each expression.

$(4x-7)(4x+7)$	$(4x+7)^2$	$(4x-7)^2$
$16x^2 - 49$	$16x^2 + 56x + 49$	$16x^2 - 56x + 49$

World View Note: There are also formulas for higher powers of binomials as well, such as $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. While French mathematician Blaise Pascal often gets credit for working with these expansions of binomials in the 17^{th} century, Chinese mathematicians had been working with them almost 400 years earlier!

3.6 Practice

Find each product and simplify your answers.

1) $(x+8)(x-8)$
2) $(a-4)(a+4)$
3) $(1+3p)(1-3p)$
4) $(x-3)(x+3)$
5) $(1-7n)(1+7n)$
6) $(8m+5)(8m-5)$
7) $(5n-8)(5n+8)$
8) $(2x+3)(2x-3)$
9) $(4x+8)(4x-8)$
10) $(b-7)(b+7)$
11) $(4y-x)(4y+x)$
12) $(7a+7b)(7a-7b)$
13) $(4m-8n)(4m+8n)$
14) $(3y-3x)(3y+3x)$
15) $(6x-2y)(6x+2y)$
16) $(1+5n)^2$
17) $(a+5)^2$
18) $(v+4)^2$
19) $(x-8)^2$
20) $(1-6n)^2$
21) $(p+7)^2$
22) $(7k-7)^2$
23) $(7-5n)^2$
24) $(4x-5)^2$
25) $(5m-8)^2$
26) $(3a+3b)^2$
27) $(5x+7y)^2$
28) $(4m-n)^2$
29) $(2x+2y)^2$
30) $(8x+5y)^2$
31) $(5+2r)^2$

32) $(m-7)^2$

33) $(2+5x)^2$

34) $(8n+7)(8n-7)$

35) $(4v-7)(4v+7)$

36) $(b+4)(b-4)$

37) $(n-5)(n+5)$

38) $(7x+7)^2$

39) $(4k+2)^2$

40) $(3a-8)(3a+8)$

3.6 Answers

1) x^2-64
2) a^2-16
3) $1-9p^2$
4) x^2-9
5) $1-49n^2$
6) $64m^2-25$
7) $25n^2-64$
8) $4x^2-9$
9) $16x^2-64$
10) b^2-49
11) $16y^2-x^2$
12) $49a^2-49b^2$
13) $16m^2-64n^2$
14) $9y^2-9x^2$
15) $36x^2-4y^2$
16) $1+10n+25n^2$
17) $a^2+10a+25$
18) $v^2+8v+16$
19) $x^2-16x+64$
20) $1-12n+36n^2$
21) $p^2+14p+49$
22) $49k^2-98k+49$
23) $49-70n+25n^2$
24) $16x^2-40x+25$
25) $25m^2-80m+64$
26) $9a^2+18ab+9b^2$
27) $25x^2+70xy+49y^2$
28) $16m^2-8mn+n^2$
29) $4x^2+8xy+4y^2$
30) $64x^2+80xy+25y^2$
31) $25+20r+4r^2$
32) $m^2-14m+49$
33) $4+20x+25x^2$

34) $64n^2 - 49$

35) $16v^2 - 49$

36) $b^2 - 16$

37) $n^2 - 25$

38) $49x^2 + 98x + 49$

39) $16k^2 + 16k + 4$

40) $9a^2 - 64$

Chapter 4: Factoring Polynomials

Section 4.1: Factoring Using the Greatest Common Factor

Objective: Find the greatest common factor of a polynomial and factor it out of the expression.

The inverse of multiplying polynomials together is factoring polynomials. There are many benefits of a polynomial being factored. We use factored polynomials to help us solve equations, learn behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials, it is very important to have very strong factoring skills.

In this lesson, we will focus on factoring using the greatest common factor or GCF of a polynomial. When we multiplied polynomials, we multiplied monomials by polynomials by distributing, such as $4x^2(2x^2-3x+8)=8x^4-12x^3+32x^2$. In this lesson, we will work the same problem backwards. For example, we will start with $8x^4-12x^3+32x^2$ and try and work backwards to the $4x^2(2x^2-3x+8)$.

To do this, we have to be able to first identify what is the GCF of a polynomial. We will introduce this idea by looking at finding the GCF of several numbers. To find the GCF of several numbers, we are looking for the largest number that can divide each number without leaving a remainder. This can often be done with quick mental math. See the example below.

Example 1. Determine the greatest common factor.

Find the GCF of 15, 24, and 27

$\frac{15}{3}=5, \frac{24}{3}=8, \frac{27}{3}=9$ Each number can be divided by 3

$\text{GCF}=3$ Our Solution

When there are variables in our problem, we can first find the GCF of the numbers using mental math. Then, we take any variables that are in common with each term, using the lowest exponent. This is shown in the next example.

Example 2. Determine the greatest common factor.

Find the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$

$\frac{24}{6}=4, \frac{18}{6}=3, \frac{12}{6}=2$ Each number can be divided by 6.

x^2y x and y are in all 3; use lowest exponents

$$\text{GCF} = 6x^2y \quad \text{Our Solution}$$

To factor out a GCF from a polynomial, we first need to identify the GCF of all the terms. This is the part that goes in front of the parentheses. Then we divide each term by the GCF, and the quotients go inside the parentheses. This is shown in the following examples.

Example 3. Factor using the greatest common factor.

$$4x^2 - 20x + 16 \quad \text{GCF is 4; divide each term by 4}$$

$$\frac{4x^2}{4} = x^2, \frac{-20x}{4} = -5x, \frac{16}{4} = 4 \quad \text{Result is what is left in parentheses}$$

$$4(x^2 - 5x + 4) \quad \text{Our Solution}$$

With factoring, we can always check our solutions by multiplying (or distributing), and the product should be the original expression.

Example 4. Factor using the greatest common factor.

$$25x^4 - 15x^3 + 20x^2 \quad \text{GCF is } 5x^2\text{; divide each term by } 5x^2$$

$$\frac{25x^4}{5x^2} = 5x^2, \frac{-15x^3}{5x^2} = -3x, \frac{20x^2}{5x^2} = 4 \quad \text{Result is what is left in parentheses}$$

$$5x^2(5x^2 - 3x + 4) \quad \text{Our Solution}$$

Example 5. Factor using the greatest common factor.

$$3x^3y^2z + 5x^4y^3z^5 - 4xy^4 \quad \text{GCF is } xy^2\text{; divide each term by } xy^2$$

$$\frac{3x^3y^2z}{xy^2} = 3x^2z, \frac{-5x^4y^3z^5}{xy^2} = 5x^3yz^5, \frac{-4xy^4}{xy^2} = -4y^2 \quad \text{Result is what is left in parentheses}$$

$$xy^2(3x^2z + 5x^3yz^5 - 4y^2) \quad \text{Our Solution}$$

World View Note: The first recorded algorithm for finding the greatest common factor comes from the Greek mathematician Euclid around the year 300 BC!

Example 6. Factor using the greatest common factor.

$$21x^3 + 14x^2 + 7x \quad \text{GCF is } 7x\text{; divide each term by } 7x$$

$$\frac{21x^3}{7x} = 3x^2, \frac{14x^2}{7x} = 2x, \frac{7x}{7x} = 1 \quad \text{Result is what is left in parentheses}$$

$$7x(3x^2 + 2x + 1) \quad \text{Our Solution}$$

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was one of the terms, dividing gave an answer of 1. Students often try to factor out the $7x$ and get zero which is incorrect. Factoring will never make terms disappear. Anything divided by itself is 1, so be sure to not forget to put the 1 into the solution.

In the next example, we will factor out the negative of the GCF. Whenever the first term of a polynomial is negative, we will factor out the negative of the GCF.

Example 7. Factor using the negative of the greatest common factor.

$-12x^5y^2+6x^4y^4-8x^3y^5$	Negative of the GCF is $-2x^3y^2$; divide each term by $-2x^3y^2$
$\frac{-12x^5y^2}{-2x^3y^2}=6x^2,\ \frac{6x^4y^4}{-2x^3y^2}=-3xy^2,\ \frac{-8x^3y^5}{-2x^3y^2}=4y^3$	Result is what is left in parentheses
$-2x^3y^2(6x^2-3xy^2+4y^3)$	Our Solution

Often the second step is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parentheses as shown in the following two examples.

Example 8. Factor using the greatest common factor.

$18a^4b^3-27a^3b^3+9a^2b^3$	GCF is $9a^2b^3$; divide each term by $9a^2b^3$
$9a^2b^3(2a^2-3a+1)$	Our Solution

Again, in the previous example, when dividing $9a^2b^3$ by itself, the answer is 1, not zero. Be very careful that each term is accounted for in your final solution.

It is possible to have a problem where the GCF is 1. If the GCF is 1, then the polynomial cannot be factored. In this case, we state that the polynomial is **prime**. This is shown in the following example.

Example 9. Factor using the greatest common factor.

$8ab-17c+49$	GCF is 1 because there are no other factors in common to all 3 terms
$8ab-17c+49$	Our Solution: Prime

4.1 Practice

Factor each polynomial using the greatest common factor. If the first term of the polynomial is negative, then factor out the negative of the greatest common factor. If the expression cannot be factored, state that it is *prime*.

1) $9+8b^2$
2) $x-5$
3) $45x^2-25$
4) $-1-2n^2$
5) $56-35p$
6) $50x-80y$
7) $8ab-35a^2b$
8) $27x^2y^5-72x^3y^2$
9) $-3a^2b+6a^3b^2$
10) $8x^3y^2+4x^3$
11) $-5x^2-5x^3-15x^4$
12) $-32n^9+32n^6+40n^5$
13) $20x^4-30x+30$
14) $21p^6+30p^2+27$
15) $20x^4-30x+30$
16) $-10x^4+20x^2+12x$
17) $30b^9+5ab-15a^2$
18) $27y^7+12y^2x+9y^2$
19) $-48a^2b^2-56a^3b-56a^5b$
20) $30m^6+15mn^2-25$
21) $20x^8y^2z^2+15x^5y^2z+35x^3y^3z$
22) $3p+12q-15q^2r^2$
23) $50x^2y+10y^2+70xz^2$
24) $30y^4z^3x^5+50y^4z^5-10y^4z^3x$
25) $30qpr-5qp+5q$
26) $28b+14b^2+35b^3+7b^5$
27) $-18n^5+3n^3-21n+3$
28) $30a^8+6a^5+27a^3+21a^2$
29) $-40x^{11}-20x^{12}+50x^{13}-50x^{14}$
30) $-24x^6-4x^4+12x^3+4x^2$
31) $-32mn^8+4m^6n+12mn^4+16mn$

4.1 Answers

1) prime
2) prime
3) $5(9x^2-5)$
4) $-1\left(1+2n^2\right)$
5) $7(8-5p)$
6) $10(5x-8y)$
7) $ab(8-35a)$
8) $9x^2y^2(3y^3-8x)$
9) $-3a^2b(1-2ab)$
10) $4x^3(2y^2+1)$
11) $-5x^2(1+x+3x^2)$
12) $-8n^5(4n^4-4n-5)$
13) $10(2x^4-3x+3)$
14) $3(7p^6+10p^2+9)$
15) $4(7m^4+10m^3+2)$
16) $-2x(5x^3-10x-6)$
17) $5(6b^9+ab-3a^2)$
18) $3y^2(9y^5+4x+3)$
19) $-8a^2b(6b+7a+7a^3)$
20) $5(6m^6+3mn^2-5)$
21) $5x^3y^2z(4x^5z+3x^2+7y)$
22) $3(p+4q-5q^2r^2)$
23) $10(5x^2y+y^2+7xz^2)$
24) $10y^4z^3(3x^5+5z^2-x)$
25) $5q(6pr-p+1)$
26) $7b(4+2b+5b^2+b^4)$
27) $-3(6n^5-n^3+7n-1)$
28) $3a^2(10a^6+2a^3+9a+7)$
29) $-10x^{11}(4+2x-5x^2+5x^3)$
30) $-4x^2(6x^4+x^2-3x-1)$
31) $-4mn(8n^7-m^5-3n^3-4)$

Section 4.2: Factoring by Grouping

Objective: Factor polynomials with four terms using grouping.

The first thing we will always do, when factoring, is try to factor out a GCF. This GCF is often a monomial. For example, in the problem $5xy+10xz$, the GCF is the monomial $5x$; so, the factored expression is $5x(y+2z)$. However, a GCF does not have to be a monomial; it could be a binomial. To see this, consider the following two examples.

Example 1. Factor completely.

$3ax-7bx$	Both have x in common; factor out x
$x(3a-7b)$	Our Solution

In the next example, we have the same type of problem as in the example above; but, instead of x, the GCF is $(2a+5b)$.

Example 2. Factor completely.

$3a(2a+5b)-7b(2a+5b)$	Both have $(2a+5b)$ in common; factor out $(2a+5b)$
$(2a+5b)(3a-7b)$	Our Solution

In Example 2, we factored out a GCF that is a binomial, $(2a+5b)$. We will use this process of factoring a binomial GCF when the original polynomial has four terms and its GCF is 1.

When we need to factor a polynomial with four terms whose GCF is 1, we will have to use another strategy to factor. We will use a process known as **grouping**. We use grouping when factoring a polynomial with four terms. Remember, factoring is the reverse of multiplying, so first we will look at a multiplication problem and then try to reverse the process.

Example 3. Multiply.

$(2a+3)(5b+2)$	Distribute $(2a+3)$ into second parentheses
$5b(2a+3)+2(2a+3)$	Distribute each monomial
$10ab+15b+4a+6$	Our Solution

The product has four terms in it. We arrived at the solution by looking at the two parts, $5b(2a+3)$ and $2(2a+3)$. When we are factoring by grouping we will always divide the problem into two parts: the first two terms and the last two terms. Then, we can factor the GCF out of both the left and right groups. When we do this, our hope is what remains in the parentheses will match on both the left term and the right term. If they match, we can pull this matching GCF out front, putting the rest in parentheses, and the expression will be factored. The next example is the same problem worked backwards, factoring instead of multiplying.

Example 4. Factor completely.

$10ab+15b+4a+6$	Split expressions into two groups
$\boxed{10ab+15b}\boxed{+4a+6}$	GCF on left is $5b$; GCF on right is 2
$\boxed{5b(2a+3)}\boxed{+2(2a+3)}$	$(2a+3)$ is the same; factor out this GCF
$(2a+3)(5b+2)$	Our Solution

The key, for grouping to work, is after the GCF is factored out of the left and right groups, the two binomials must match exactly. If there is any difference between the two, we either have to do some adjusting or it cannot be factored using the grouping method. Consider the following example.

Example 5. Factor completely.

$6x^2+9xy-14x-21y$	Split expression into two groups
$\boxed{6x^2+9xy}\boxed{-14x-21y}$	GCF on left is $3x$; GCF on right is 7
$\boxed{3x(2x+3y)}\boxed{+7(-2x-3y)}$	The signs in the parentheses don't match!

When the signs don't match in both terms, we can easily make them match by factoring the negative of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second parentheses. In general, if the third term of the four - term expression is subtracted, then factor out the negative of the GCF for the second group of two terms.

$\boxed{3x(2x+3y)}\boxed{-7(2x+3y)}$	$(2x+3y)$ is the same; factor out this GCF
$(2x+3y)(3x-7)$	Our Solution

Often we can recognize early that we need to use the negative of the GCF when factoring. If the first term of the first binomial is positive in the problem, we will also want the first term of the second binomial to be positive. If the first term of the second binomial is negative, then we will use the negative of the GCF to be sure they match.

Example 6. Factor completely.

$5xy-8x-10y+16$	Split expression into two groups
$\boxed{5xy-8x}\boxed{-10y+16}$	GCF on left is x; GCF on right is -2
$\boxed{x(5y-8)}\boxed{-2(5y-8)}$	$(5y-8)$ is the same; factor out this GCF
$(5y-8)(x-2)$	Our Solution

Sometimes when factoring the GCF out of the left or right group there is no GCF to factor out other than one. In this case we will use either the GCF of 1 or -1. Often this is all we need to be sure the two binomials match.

Example 7. Factor completely.

$12ab-14a-6b+7$	Split expression into two groups
$\boxed{12ab-14a}\boxed{-6b+7}$	GCF on left is $2a$; negative of GCF on right is -1
$\boxed{2a(6b-7)}\boxed{-1(6b-7)}$	$(6b-7)$ is the same; factor out this GCF
$(6b-7)(2a-1)$	Our Solution

Example 8. Factor completely.

$6x^3-15x^2+2x-5$	Split expression into two groups
$\boxed{6x^3-15x^2}\boxed{+2x-5}$	GCF on left is $3x^2$; GCF on right is 1
$\boxed{3x^2(2x-5)}\boxed{+1(2x-5)}$	$(2x-5)$ is the same; factor out this GCF
$(2x-5)(3x^2+1)$	Our Solution

Example 9. Factor completely.

$12x^3+6x^2+5x+10$	Split expression into two groups
$\boxed{12x^3+6x^2}\boxed{+5x+10}$	GCF on left is $6x^2$; GCF on right is 5
$\boxed{6x^2(2x+1)}\boxed{+5(x+2)}$	Factors are not the same. This will happen even if we rearrange the terms. Therefore, it cannot be factored by grouping.
$12x^3+6x^2+5x+10$	Our Solution, Prime

4.2 Practice

Factor each expression completely.

1) $40r^3 - 8r^2 - 25r + 5$
2) $35x^3 - 10x^2 - 56x + 16$
3) $3n^3 - 2n^2 - 9n + 6$
4) $14v^3 + 10v^2 - 7v - 5$
5) $15b^3 + 21b^2 - 35b - 49$
6) $6x^3 - 48x^2 + 5x - 40$
7) $3x^3 + 15x^2 + 2x + 10$
8) $9x^3 + 3x^2 + 4x + 8$
9) $35x^3 - 28x^2 - 20x + 16$
10) $7n^3 + 21n^2 - 5n - 15$
11) $7xy - 49x + 5y - 35$
12) $42r^3 - 49r^2 + 18r - 21$
13) $32xy + 40x^2 + 12y + 15x$
14) $15ab - 6a + 5b^3 - 2b^2$
15) $16xy - 56x + 2y - 7$
16) $3mn - 8m + 15n - 40$
17) $x^3 - 5x^2 + 7x - 21$
18) $5mn + 2m - 25n - 10$
19) $40xy + 35x - 8y^2 - 7y$
20) $6a^2 + 3a - 4b^2 + 2b$
21) $32uv - 20u + 24v - 15$
22) $4uv + 14u^2 + 12v + 42u$
23) $10xy + 25x + 12y + 30$
24) $24xy - 20x - 30y^3 + 25y^2$
25) $3uv - 6u^2 - 7v + 14u$
26) $56ab - 49a - 16b + 14$
27) $2xy - 8x^2 + 7y^3 - 28y^2x$
28) $28p^3 + 21p^2 + 20p + 15$
29) $16xy - 6x^2 + 8y - 3x$
30) $8xy + 56x - y - 7$

4.2 Answers

1) $(8r^2-5)(5r-1)$
2) $(5x^2-8)(7x-2)$
3) $(n^2-3)(3n-2)$
4) $(2v^2-1)(7v+5)$
5) $(3b^2-7)(5b+7)$
6) $(6x^2+5)(x-8)$
7) $(3x^2+2)(x+5)$
8) Prime
9) $(7x^2-4)(5x-4)$
10) $(7n^2-5)(n+3)$
11) $(7x+5)(y-7)$
12) $(7r^2+3)(6r-7)$
13) $(8x+3)(4y+5x)$
14) $(3a+b^2)(5b-2)$
15) $(8x+1)(2y-7)$
16) $(m+5)(3n-8)$
17) Prime
18) $(m-5)(5n+2)$
19) $(5x-y)(8y+7)$
20) Prime
21) $(4u+3)(8v-5)$
22) $2(u+3)(2v+7u)$
23) $(5x+6)(2y+5)$
24) $(4x-5y^2)(6y-5)$
25) $(3u-7)(v-2u)$
26) $(7a-2)(8b-7)$
27) $(2x+7y^2)(y-4x)$
28) $(7p^2+5)(4p+3)$
29) $(2x+1)(8y-3x)$
30) $(8x-1)(y+7)$

Section 4.3: Factoring Trinomials When the Leading Coefficient is One

Objective: Factor trinomials when the leading coefficient or the coefficient of x^2 is 1.

Factoring polynomials with three terms, or factoring trinomials, is the most important type of factoring to be mastered. Since factoring can be thought of as the reverse of multiplication, we will start with a multiplication problem and look at how we can reverse the process.

Example 1. Multiply and simplify.

$(x+6)(x-4)$	Distribute $(x+6)$ through second parentheses
$x(x+6)-4(x+6)$	Distribute each monomial through parentheses
$x^2+6x-4x-24$	Combine like terms
$x^2+2x-24$	Our Solution

You may notice that if you reverse the last three steps, the process looks like grouping. This is because it is grouping! The GCF of the left two terms is x and the negative of the GCF of the second two terms is -4. The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This is shown in the following example, which is the same problem worked backwards:

Example 2. Factor completely.

$x^2+2x-24$	Split middle term into $+6x-4x$
$x^2+6x-4x-24$	Grouping; GCF on left is x; negative of GCF on right is -4
$x(x+6)-4(x+6)$	$(x+6)$ is the same; factor out this GCF
$(x+6)(x-4)$	Our Solution

The trick to making these problems work is in how we split the middle term. Why did we pick $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that works! So, how do we know what is the one combination that works? To find the correct way to split the middle term, we find a pair of numbers that multiply to obtain the last term in the trinomial and also sum to the number that is the coefficient of the middle term of the trinomial. In the previous example that would mean we wanted to multiply to -24 and sum to 2. The only numbers that can do this are 6 and -4 ($6\cdot -4=-24$ and $6+(-4)=2$). This process is shown in the next few examples.

Example 3. Factor completely.

$x^2+9x+18$	Multiply to 18; sum to 9
$x^2+6x+3x+18$	6 and 3; split the middle term
$x(x+6)+3(x+6)$	Factor by grouping
$(x+6)(x+3)$	Our Solution

Example 4. Factor completely.

x^2-4x+3	Multiply to 3; sum to -4
$x^2-3x-x+3$	-3 and -1; split the middle term
$x(x-3)-1(x-3)$	Factor by grouping
$(x-3)(x-1)$	Our Solution

Example 5. Factor completely.

$x^2-8x-20$	Multiply to -20; sum to -8
$x^2-10x+2x-20$	-10 and 2; split the middle term
$x(x-10)+2(x-10)$	Factor by grouping
$(x-10)(x+2)$	Our Solution

Often when factoring, we have two variables. These problems solve just like problems with one variable, using the coefficients to decide how to split the middle term.

Example 6. Factor completely.

$a^2-9ab+14b^2$	Multiply to 14; sum to -9
$a^2-7ab-2ab+14b^2$	-7 and -2; split the middle term
$a(a-7b)-2b(a-7b)$	Factor by grouping
$(a-7b)(a-2b)$	Our Solution

Warning! Notice that it is very important to be aware of negatives, as we find the pair of numbers we will use to split the middle term. Consider the following example, done incorrectly, ignoring negative signs:

Factor x^2+5x-6	Multiply to 6; sum to 5
$x^2+2x+3x-6$	2 and 3; split the middle term
$x(x+2)+3(x-2)$	Factor by grouping
???	Binomials do not match!

Because we did not use the negative sign with the 6 to find our pair of numbers, the binomials did not match and grouping was not able to work at the end. Now the problem will be done correctly:

Example 7. Factor completely.

x^2+5x-6	Multiply to -6; sum to 5
$x^2+6x-1x-6$	6 and -1; split the middle term
$x(x+6)-1(x+6)$	Factor by grouping
$(x+6)(x-1)$	Our Solution

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1; our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it, however, when we have no number (technically we have a 1) in front of x^2. In all of the problems we have factored in this lesson, there is no number written in front of x^2. If this is the case, then we can use this shortcut. This is shown in the next few examples.

Example 8. Factor completely.

$x^2-7x-18$	Multiply to -18; sum to -7
	-9 and 2; write the factors
$(x-9)(x+2)$	Our Solution

Example 9. Factor completely.

$m^2-mn-30n^2$	Multiply to -30; sum to -1
	5 and -6; write the factors; don't forget the second variable
$(m+5n)(m-6n)$	Our Solution

It is possible to have a problem that does not factor. If there is no combination of numbers that multiplies and adds up to the correct numbers, then we say we cannot factor the polynomial or we say the polynomial is prime. This is shown in the following example.

Example 10. Factor completely.

x^2+2x+6	Multiply to 6; sum to 2
$1\cdot 6$ and $2\cdot 3$	Only possibilities to multiply to 6; none sum to 2
Prime	Our Solution

When factoring, it is important not to forget about the GCF. If all of the terms in a problem have a common factor, we will want to first factor out the GCF before we attempt using any other method. The next three examples illustrate this technique:

Example 11. Factor completely.

$3x^2-24x+45$	GCF of all terms is 3; factor out 3
$3\left(x^2-8x+15\right)$	Multiply to 15; sum to -8
	-5 and -3; write the factors
$3\left(x^2-8x+15\right)$	
$3(x-5)(x-3)$	Our Solution

Example 12. Factor completely.

$4x^2y-8xy-32y$	GCF of all terms is $4y$; factor out $4y$
$4y(x^2-2x-8)$	Multiply to -8; sum to -2
	-4 and 2; write the factors
$4y(x-4)(x+2)$	Our Solution

Example 13. Factor completely.

$7a^4b^2+28a^3b^2-35a^2b^2$	GCF of all terms is $7a^2b^2$; factor out $7a^2b^2$
$7a^2b^2(a^2+4a-5)$	Multiply to -5; sum to 4
	-1 and 5; write the factors
$7a^2b^2(a-1)(a+5)$	Our Solution

Again it is important to comment on the shortcut of jumping right to the factors. This only works if there is no written coefficient of x^2; that is, the leading coefficient is understood to be 1. Be careful not to use this shortcut on all factoring problems!

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables we are solving for, just as codes used letters to represent an unknown message.

4.3 Practice

Factor each expression completely.

1) $p^2+17p+72$
2) x^2+x-72
3) n^2-9n+8
4) x^2+x-30
5) $x^2-9x-10$
6) $x^2+13x+40$
7) $b^2+12b+32$
8) $b^2-17b+70$
9) $x^2+3x-70$
10) $x^2+3x-18$
11) $n^2-8n+15$
12) $a^2-6a-27$
13) $p^2+15p+54$
14) $p^2+7p-30$
15) c^2-4c+9
16) $m^2-15mn+50n^2$
17) $u^2-8uv+15v^2$
18) $m^2-3mn-40n^2$
19) $m^2+2mn-8n^2$
20) $x^2+10xy+16y^2$
21) $x^2-11xy+18y^2$
22) $u^2-9uv+14v^2$
23) $x^2+xy-12y^2$
24) $x^2+14xy+45y^2$
25) $x^2+4xy-12y^2$
26) $4x^2+52x+168$
27) $5a^2+60a+100$
28) $7w^2+5w-35$
29) $6a^2+24a-192$
30) $5v^2+20v-25$
31) $6x^2+18xy+12y^2$

32) $5m^2+30mn-80n^2$

33) $6x^2+96xy+378y^2$

34) $6m^2-36mn-162n^2$

35) $n^2-15n+56$

36) $5n^2-45n+40$

4.3 Answers

1) $(p+9)(p+8)$
2) $(x-8)(x+9)$
3) $(n-8)(n-1)$
4) $(x-5)(x+6)$
5) $(x+1)(x-10)$
6) $(x+5)(x+8)$
7) $(b+8)(b+4)$
8) $(b-10)(b-7)$
9) $(x-7)(x+10)$
10) $(x-3)(x+6)$
11) $(n-5)(n-3)$
12) $(a+3)(a-9)$
13) $(p+6)(p+9)$
14) $(p+10)(p-3)$
15) Prime
16) $(m-5n)(m-10n)$
17) $(u-5v)(u-3v)$
18) $(m+5n)(m-8n)$
19) $(m+4n)(m-2n)$
20) $(x+8y)(x+2y)$
21) $(x-9y)(x-2y)$
22) $(u-7v)(u-2v)$
23) $(x-3y)(x+4y)$
24) $(x+5y)(x+9y)$
25) $(x+6y)(x-2y)$
26) $4(x+7)(x+6)$
27) $5(a+10)(a+2)$
28) Prime
29) $6(a-4)(a+8)$
30) $5(v-1)(v+5)$
31) $6(x+2y)(x+y)$
32) $5(m-2n)(m+8n)$
33) $6(x+9y)(x+7y)$
34) $6(m-9n)(m+3n)$
35) $(n-8)(n-7)$

36) $5(n-8)(n-1)$

Section 4.4: Factoring Special Forms of Polynomials

Objective: Identify and factor special forms of polynomials including a difference of two squares and perfect square trinomials.

When factoring, there are a few special forms of polynomials that, if we can recognize them, help us factor polynomials. The first is one we have seen before. When multiplying special products, we found that a sum and a difference could multiply to a difference of two squares. Here, we will use this special product to help us factor the difference of two squares.

Difference of Two Squares: $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

If we are subtracting two squares, then the expression will always factor to the sum and difference of the square roots.

Example 1. Factor completely.

x^2-16	Subtracting two squares; the square roots are x and 4
$(x+4)(x-4)$	Our Solution

Example 2. Factor completely.

$9a^2-25b^2$	Subtracting two squares; the square roots are $3a$ and $5b$
$(3a+5b)(3a-5b)$	Our Solution

In the next example, we will see that, generally, the sum of two squares cannot be factored.

Example 3. Factor completely.

x^2+36	No bx term; we use $0x$.
$x^2+0x+36$	Multiply to 36; sum to 0
$1\cdot 36,\ 2\cdot 18,\ 3\cdot 12,\ 4\cdot 9,\ 6\cdot 6$	No combinations that multiplies to 36 and sum to 0
Prime (cannot be factored)	Our Solution

It turns out that a sum of two squares is generally considered to be prime when the exponent is 2. If the exponent is greater than 2, then factoring the sum of two squares will go beyond the scope of this course.

Sum of Two Squares: $\boldsymbol{a^2+b^2}$ **= prime (generally cannot be factored)**

A great example where we see a sum of two squares and a difference of two squares together would be factoring a difference of fourth powers. Because the square root of a fourth power is a square $\left(\sqrt{a^4}=a^2\right)$, we can factor a difference of fourth powers, just like we factor a difference of two squares, to a sum and difference of the square roots. This will give us two factors: one which will be a prime sum of two squares; and a second that will be a difference of two squares, which we can factor again. This is shown in the following two examples.

Example 4. Factor completely.

a^4-b^4	Difference of two squares with square roots a^2 and b^2
$(a^2+b^2)(a^2-b^2)$	The first factor is prime; the second is a difference of two squares with square roots a and b
$(a^2+b^2)(a+b)(a-b)$	Our Solution

Example 5. Factor completely.

x^4-16	Difference of two squares with square roots x^2 and 4
$(x^2+4)(x^2-4)$	The first factor is prime; the second is a difference of two squares with square roots x and 2
$(x^2+4)(x+2)(x-2)$	Our Solution

Another factoring formula is the perfect square trinomial. We had a shortcut for squaring binomials, which can be reversed to help us factor a perfect square trinomial.

Perfect Square Trinomial: $\boldsymbol{a^2+2ab+b^2=(a+b)^2}$

Here is how to recognize a perfect square trinomial:

(1) The first term is a square of a monomial or an integer.
(2) The middle term is two times the product of the square root of the first and last terms.
(3) The third term is a square of a monomial or an integer.

Then, we can factor a perfect square trinomial using the square roots of the first and last terms and the sign from the middle term. This is shown in the following examples.

Example 6. Factor completely.

x^2-6x+9	$x^2=(x)^2;\,6x=2(x)(3);\,9=(3)^2$
$(x)^2-2\cdot x\cdot 3+(3)^2$	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$(x-3)^2$	Our Solution

Example 7. Factor completely.

$4x^2+20xy+25y^2$	$4x^2=(2x)^2;\,20xy=2(2x)(5y);\,25y^2=(5y)^2$
$(2x)^2+2\cdot 2x\cdot 5y+(5y)^2$	Perfect square trinomial; use square roots from first and last terms and sign from the middle

$(2x+5y)^2$ Our Solution

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. For example, problems would be written as: "three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou." If g represents a sheaf of good crop, m represents a sheaf of mediocre crop, and b respresents a sheaf of bad crop, then we can say that the polynomial $3g+2m+b$ equals 29.

The following table sumarizes all of the formulas that we can use to factor special forms of polynomials.

Factoring Special Forms of Polynomials

Difference of Two Squares:	$a^2-b^2=(a+b)(a-b)$
Sum of Two Squares:	$a^2+b^2=$ Prime (generally cannot be factored)
Perfect Square Trinomial:	$a^2+2ab+b^2=(a+b)^2$

As always, when factoring special forms of polynomials, it is important to check for a GCF first. Only after checking for a GCF should we use the special products used in the factoring formulas. This is shown in the following examples.

Example 8. Factor completely.

$72x^2-2$ GCF is 2

$2(36x^2-1)$ Difference of two squares; square roots are $6x$ and 1

$2(6x+1)(6x-1)$ Our Solution

Example 9. Factor completely.

$48x^2y-24xy+3y$ GCF is $3y$

$3y(16x^2-8x+1)$ $16x^2=(4x)^2; 8x=2(4x)(1); 1=(1)^2$; perfect square trinomial

$16x^2-8x+1=(4x)^2-2\cdot 4x\cdot 1+(1)^2$

Use square roots from first and last terms and sign from the middle

$3y(4x-1)^2$ Our Solution

4.4 Practice

Factor each expression completely.

1) $r^2 - 16$
2) $x^2 - 9$
3) $v^2 - 25$
4) $x^2 - 1$
5) $p^2 - 4$
6) $4v^2 - 1$
7) $9k^2 - 4$
8) $9a^2 - 1$
9) $3x^2 - 27$
10) $5n^2 - 20$
11) $16x^2 - 36$
12) $125x^2 + 45y^2$
13) $18a^2 - 50b^2$
14) $4m^2 + 64n^2$
15) $a^2 - 2a + 1$
16) $k^2 + 4k + 4$
17) $x^2 + 6x + 9$
18) $n^2 - 8n + 16$
19) $x^2 - 6x + 9$
20) $k^2 - 4k + 4$
21) $25p^2 - 10p + 1$
22) $x^2 + 2x + 1$
23) $25a^2 + 30ab + 9b^2$
24) $x^2 + 8xy + 16y^2$
25) $4a^2 - 20ab + 25b^2$
26) $49x^2 + 36y^2$
27) $8x^2 - 24xy + 18y^2$
28) $20x^2 + 20xy + 5y^2$
29) $a^4 - 81$
30) $x^4 - 256$
31) $16 - z^4$
32) $n^4 - 1$

33) $x^4 - y^4$

34) $16a^4 - b^4$

35) $m^4 - 81b^4$

36) $81c^4 - 16d^4$

37) $18m^2 - 24mn + 8n^2$

38) $w^4 + 225$

4.4 Answers

1) $(r+4)(r-4)$
2) $(x+3)(x-3)$
3) $(v+5)(v-5)$
4) $(x+1)(x-1)$
5) $(p+2)(p-2)$
6) $(2v+1)(2v-1)$
7) $(3k+2)(3k-2)$
8) $(3a+1)(3a-1)$
9) $3(x+3)(x-3)$
10) $5(n+2)(n-2)$
11) $4(2x+3)(2x-3)$
12) $5(25x^2+9y^2)$
13) $2(3a+5b)(3a-5b)$
14) $4(m^2+16n^2)$
15) $(a-1)^2$
16) $(k+2)^2$
17) $(x+3)^2$
18) $(n-4)^2$
19) $(x-3)^2$
20) $(k-2)^2$
21) $(5p-1)^2$
22) $(x+1)^2$
23) $(5a+3b)^2$
24) $(x+4y)^2$
25) $(2a-5b)^2$
26) Prime
27) $2(2x-3y)^2$
28) $5(2x+y)^2$
29) $(a^2+9)(a+3)(a-3)$
30) $(x^2+16)(x+4)(x-4)$
31) $(4+z^2)(2+z)(2-z)$
32) $(n^2+1)(n+1)(n-1)$
33) $(x^2+y^2)(x+y)(x-y)$

34) $(4a^2 + b^2)(2a + b)(2a - b)$

35) $(m^2 + 9b^2)(m + 3b)(m - 3b)$

36) $(9c^2 + 4d^2)(3c + 2d)(3c - 2d)$

37) $2(3m - 2n)^2$

38) Prime

Section 4.5: A General Factoring Strategy

Objective: Identify and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which strategy to use when. Here, we will attempt to organize all the different factoring methods we have seen. A large part of deciding how to solve a problem is based on how many terms are in the problem. For all problem types, we will always try to factor out the GCF first.

Factoring Strategy (GCF First!!!!!)

- **2 terms**: sum or difference of two squares:
  $a^2 - b^2 = (a+b)(a-b)$
  $a^2 + b^2 =$ prime (generally cannot be factored)
- **3 terms**: Watch for trinomials with leading coefficient of one and perfect square trinomials!
  $a^2 + 2ab + b^2 = (a+b)^2$
- **4 terms**: grouping

We will use the above strategy to factor each of the following examples. Here, the emphasis will be on which strategy to use rather than the steps used in that method.

Example 1. Factor completely.

$x^2 - 23x + 42$	GCF=1; so nothing to factor out of all three terms
	Three terms; multiply to 42; sum to -23
	-2 and -21; write the factors
$(x-2)(x-21)$	Our Solution

Example 2. Factor completely.

$z^2 + 6z - 9$	GCF=1; so nothing to factor out of all three terms
	Three terms; multiply to -9; sum to 6
	Factors of -9: $(-1)(9), (1)(-9), (3)(-3)$; none sum to 6
Prime (cannot be factored)	Our Solution

Example 3. Factor completely.

$4x^2 + 56xy + 196y^2$	GCF first; factor out 4 from each term
$4(x^2 + 14xy + 49y^2)$	Three terms, $x^2 = (x)^2; 14xy = 2(x)(7y); 49y^2 = (7y)^2$
	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$4(x+7y)^2$	Our Solution

Example 4. Factor completely.

$5x^2y+15xy-35x^2-105x$	GCF first; factor out $5x$ from each term
$5x(xy+3y-7x-21)$	Four terms; try grouping
$5x[y(x+3)-7(x+3)]$	$(x+3)$ match
$5x(x+3)(y-7)$	Our Solution

Example 5. Factor completely.

$100x^2-400$	GCF first; factor out 100 from each term
$100(x^2-4)$	Two terms; difference of two squares
$100(x+2)(x-2)$	Our Solution

Example 6. Factor completely.

$108x^3y^2-36x^2y^2+3xy^2$	GCF first; factor out $3xy^2$ from each term
$3xy^2(36x^2-12x+1)$	Three terms; $36x^2=(6x)^2; 12x=2(6x)(1); 1=(1)^2$
	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$3xy^2(6x-1)^2$	Our Solution

World View Note: Variables originated in ancient Greece where Aristotle would use a single capital letter to represent a number.

It is important to be comfortable and con dent not just with using all the factoring methods, but also with deciding on which method to use. This is why your practice with these problems is very important!

4.5 Practice

Factor each expression completely.

1) $16x^2+48xy+36y^2$
2) n^2-n
3) $x^2-4xy+3y^2$
4) $45u^2-150uv+125v^2$
5) $64x^2+49y^2$
6) m^2-4n^2
7) $3m^3-6m^2n-24n^2m$
8) $2x^3+6x^2y-20y^2x$
9) n^3+7n^2+10n
10) $16a^2-9b^2$
11) $5x^2+2x$
12) $2x^2-10x+12$
13) $3k^3-27k^2+60k$
14) $32x^2-18y^2$
15) $16x^2-8xy+y^2$
16) v^2+v
17) $27m^2-48n^2$
18) x^3+4x^2
19) $9n^3-3n^2$
20) $2m^2+6mn-20n^2$
21) $16x^2+1$
22) $9x^2-25y^2$
23) $mn+3m-4xn-12x$
24) $24az-18ah+60yz-45yh$
25) $20uv-60u^3-5xv+15xu^2$
26) $36b^2c-24b^2d+24xc-16xd$

4.5 Answers

1) $4(2x+3y)^2$
2) $n(n-1)$
3) $(x-3y)(x-y)$
4) $5(3u-5v)^2$
5) Prime
6) $(m+2n)(m-2n)$
7) $3m(m+2n)(m-4n)$
8) $2x(x+5y)(x-2y)$
9) $n(n+2)(n+5)$
10) $(4a+3b)(4a-3b)$
11) $x(5x+2)$
12) $2(x-2)(x-3)$
13) $3k(k-5)(k-4)$
14) $2(4x+3y)(4x-3y)$
15) $(4x-y)^2$
16) $v(v+1)$
17) $3(3m+4n)(3m-4n)$
18) $x^2(x+4)$
19) $3n^2(3n-1)$
20) $2(m-2n)(m+5n)$
21) Prime
22) $(3x+5y)(3x-5y)$
23) $(m-4x)(n+3)$
24) $3(2a+5y)(4z-3h)$
25) $5(4u-x)(v-3u^2)$
26) $4(3b^2+2x)(3c-2d)$

Section 4.6: Solving Equations by Factoring

Objective: Solve quadratic equations by factoring and using the zero product rule.

When solving linear equations such as $2x-5=21$; we can solve for the variable directly by adding 5 and dividing by 2, on both sides, to get 13. However, when we have x^2 (or a higher power of x), we cannot just isolate the variable, as we did with the linear equations. One method that we can use to solve for the variable is known as the zero product rule.

Zero Product Rule: If $\boldsymbol{a \cdot b = 0}$, then either $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$.

The zero product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero. We can use this to help us solve factored polynomials as in the following example.

Example 1. Solve the equation.

$$(2x-3)(5x+1)=0 \quad \text{One factor must be zero}$$

$$2x-3=0 \quad \text{or} \quad 5x+1=0 \quad \text{Set each factor equal to zero}$$

$$\underline{\;+3\;\;+3\;} \qquad \underline{\;-1\;\;-1\;} \quad \text{Solve each equation}$$

$$\frac{2x}{2}=\frac{3}{2} \quad \text{or} \quad \frac{5x}{5}=\frac{-1}{5}$$

$$x=\frac{3}{2} \quad \text{or} \quad \frac{-1}{5} \quad \text{Our Solution}$$

For the zero product rule to work, we must have factors to set equal to zero. This means if the problem is not already factored, we will need to factor it first, if at all possible.

Example 2. Solve the equation by factoring.

$$x^2-7x+12=0 \quad \text{Multiply to 12; sum to } -7$$

$$(x-3)(x-4)=0 \quad \text{Numbers are } -3 \text{ and } -4.$$

$$x-3=0 \quad \text{or} \quad x-4=0 \quad \text{Since one factor must be zero, set each factor equal to zero}$$

$$\underline{\;+3\;\;+3\;} \qquad \underline{\;+4\;\;+4\;} \quad \text{Solve each equation}$$

$$x=3 \quad \text{or} \quad x=4 \quad \text{Our Solution}$$

Another important part of the zero product rule is that before we factor, one side of the equation must be zero. If one side of the equation is not zero, we must move terms around so that one side of the equation is zero. Generally, we like the x^2 term to be positive.

Example 3. Solve the equation by factoring.

$x^2 = 8x - 15$	Set equal to zero by moving terms to the left
$\underline{-8x+15 \quad -8x+15}$	
$x^2 - 8x + 15 = 0$	Factor; multiply to 15; sum to -8
$(x-5)(x-3) = 0$	Numbers are -5 and -3
$x - 5 = 0$ or $x - 3 = 0$	Set each factor equal to zero
$\underline{+5 \quad +5} \qquad \underline{+3 \quad +3}$	Solve each equation
$x = 5$ or $x = 3$	Our Solutions

Example 4. Solve the equation by factoring.

$(x-7)(x+3) = -9$	Not equal to zero; multiply first; use FOIL
$x^2 - 7x + 3x - 21 = -9$	Combine like terms
$x^2 - 4x - 21 = -9$	Move -9 to other side so equation equals zero
$\underline{+9 \quad +9}$	
$x^2 - 4x - 12 = 0$	Factor; multiply to -12; sum to -4
$(x-6)(x+2) = 0$	Numbers are -6 and $+2$
$x - 6 = 0$ or $x + 2 = 0$	Set each factor equal to zero
$\underline{+6 \quad +6} \qquad \underline{-2 \quad -2}$	Solve each equation
$x = 6$ or $x = -2$	Our Solution

Example 5. Solve the equation by factoring.

$3x^2 + 4x - 5 = 7x^2 + 4x - 14$	Set equal to zero by moving terms to the right side of the equal sign
$\underline{-3x^2 - 4x + 5 \quad -3x^2 - 4x + 5}$	
$0 = 4x^2 - 9$	Factor using the difference of two squares
$0 = (2x+3)(2x-3)$	One factor must be zero
$2x + 3 = 0$ or $2x - 3 = 0$	Set each factor equal to zero
$\underline{-3 \quad -3} \qquad \underline{+3 \quad +3}$	Solve each equation
$\frac{2x}{2} = \frac{-3}{2}$ or $\frac{2x}{2} = \frac{3}{2}$	
$x = -\frac{3}{2}$ or $\frac{3}{2}$	Our Solutions

Most problems with x^2 will have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 6. Solve the equation by factoring.

$4x^2 = 12x - 9$ $\underline{-12x+9 \quad -12x+9}$	Set equal to zero by moving terms to the left
$4x^2 - 12x + 9 = 0$	Factor; $4x^2 = (2x)^2; 12x = 2(2x)(3); 9 = (3)^2$
$(2x-3)^2 = 0$	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$2x - 3 = 0$	Set this factor equal to zero
$\underline{+3 \quad +3}$ $\frac{2x}{2} = \frac{3}{2}$	Solve the equation
$x = \frac{3}{2}$	Our Solution

As always, it will be important to factor out the GCF first, if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This may give us more than just two solutions. The next few examples illustrate this.

Example 7. Solve the equation by factoring.

$4x^2 = 8x$ $\underline{-8x \quad -8x}$	Set equal to zero by moving the $8x$ to the left side of the equal sign
$4x^2 - 8x = 0$	
$4x(x-2) = 0$	One factor must be zero
$\frac{4x}{4} = \frac{0}{4}$ or $x - 2 = 0$	Set each factor equal to zero
$\underline{+2 \quad +2}$	Solve each equation
$x = 0$ or 2	Our Solution

Example 8. Solve the equation by factoring.

$2x^3 - 14x^2 + 24x = 0$	Factor out the GCF of $2x$
$2x(x^2 - 7x + 12) = 0$	Multiply to 12; sum to -7
$2x(x-3)(x+4) = 0$	Numbers are -3 and -4
$\frac{2x}{2} = \frac{0}{2}$ or $x - 3 = 0$ or $x - 4 = 0$	Set each factor equal to zero
$\underline{+3 \quad +3}$ $\quad$ $\underline{+4 \quad +4}$	Solve each equation
$x = 0$ or 3 or 4	Our Solutions

Example 9. Solve the equation by factoring.

$$3x^2+3x-60=0 \quad \text{Factor out the GCF of 3}$$
$$3(x^2+x-20)=0 \quad \text{Multiply to } -20\text{; sum to 1}$$
$$3(x+5)(x-4)=0 \quad \text{Numbers are 5 and } -4$$
$$3(x+5)\,(x-4) \quad \text{One factor must be zero}$$
$$3=0 \quad \text{or} \quad x+5=0 \quad \text{or} \quad x-4=0 \quad \text{Set each factor equal to zero}$$
$$3\neq 0 \qquad \underline{-5\quad -5} \qquad \underline{+4\quad +4} \quad \text{Solve each equation}$$
$$x=-5 \quad \text{or} \quad x=4$$
$$x=-5 \quad \text{or} \quad x=4 \quad \text{Our Solutions}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we get a false statement. No solution comes from this factor. Often a student will skip setting the GCF factor equal to zero if there are no variables in the GCF, which is acceptable.

Just as not all polynomials can be factored, not all equations can be solved by factoring. If an equation does not factor we will have to solve it using another method. These other methods are saved for another course.

World View Note: While factoring works great to solve problems with x^2, Tartaglia, in 16th century Italy, developed a method to solve problems with x^3. He kept his method a secret until another mathematician, Cardano, talked him out of his secret and published the results. To this day the formula is known as Cardano's Formula.

A question often asked is if it is possible to get rid of the square on the variable by taking the square root of both sides. While it is possible, there are a few properties of square roots that will not be covered in this course, and thus it is common to break a rule of roots that we are not aware of at this point. The short reason we want to avoid this for now is because taking a square root will only allow us to consider the principal or positive square root. For now, **never** take the square root of both sides!

4.6 Practice

Solve each equation by factoring.

1) $(k-7)(k+2)=0$
2) $(a+4)(a-3)=0$
3) $(x-1)(x+4)=0$
4) $0=(2x+5)(x-7)$
5) $6x^2-150=0$
6) $p^2+4p-32=0$
7) $2n^2+10n-28=0$
8) $m^2-m-30=0$
9) $x^2-4x-8=-8$
10) $v^2-8v-3=-3$
11) $x^2-5x-1=-5$
12) $a^2-6a+6=-2$
13) $7r^2+84=-49r$
14) $7m^2-224=28m$
15) $x^2-6x=16$
16) $7n^2-28n=0$
17) $3v^2=5v$
18) $2b^2=-3b$
19) $9x^2=30x-25$
20) $3n^2+39n=-36$
21) $4k^2+18k-23=6k-7$
22) $a^2+7a-9=-3+6a$
23) $9x^2-46+7x=7x+8x^2+3$
24) $x^2+10x+30=6$
25) $2m^2+19m+40=-5m$
26) $5n^2+45n+38=-2$
27) $5x(3x-6)-5x^2=x^2+6x+45$
28) $8x^2+11x-48=3x$
29) $41p^2+183p-196=183p+5p^2$
30) $121w^2+8w-7=8w-6$

4.6 Answers

1) $7, -2$
2) $-4, 3$
3) $1, -4$
4) $-\frac{5}{2}, 7$
5) $-5, 5$
6) $4, -8$
7) $2, -7$
8) $-5, 6$
9) $0, 4$
10) 0.8
11) $1, 4$
12) $2, 4$
13) $-4, -3$
14) $-4, 8$
15) $-2, 8$
16) $0, 4$
17) $0, \frac{5}{3}$
18) $-\frac{3}{2}, 0$
19) $\frac{5}{3}$
20) $-12, -1$
21) $-4, 1$
22) $2, -3$
23) $-7, 7$
24) $-4, -6$
25) $-10, -2$
26) $-8, -1$
27) $-1, 5$
28) $-3, 2$
29) $-\frac{7}{3}, \frac{7}{3}$
30) $-\frac{1}{11}, \frac{1}{11}$

Chapter 5: Organizing Data

Section 5.1: Introduction

Objective: Recognize types of data.

The term "statistics" emerged around the 18th century due to the need of governments to collect demographic and economic data. Nowadays, some of the fields that use statistics include finance, economics, actuarial science, biostatistics, etc.

Definition. Statistics *is the art and science of collecting, analyzing, presenting, and interpreting data. It provides tools for predicting and forecasting the use of data through statistical models.*

What are Data?

Data are observations that have been collected. Examples include measurements, income levels, or responses to survey questions. Any set of data contains information about some group of individuals or things. The information is organized in variables.

A variable is a characteristic of an individual or some thing.

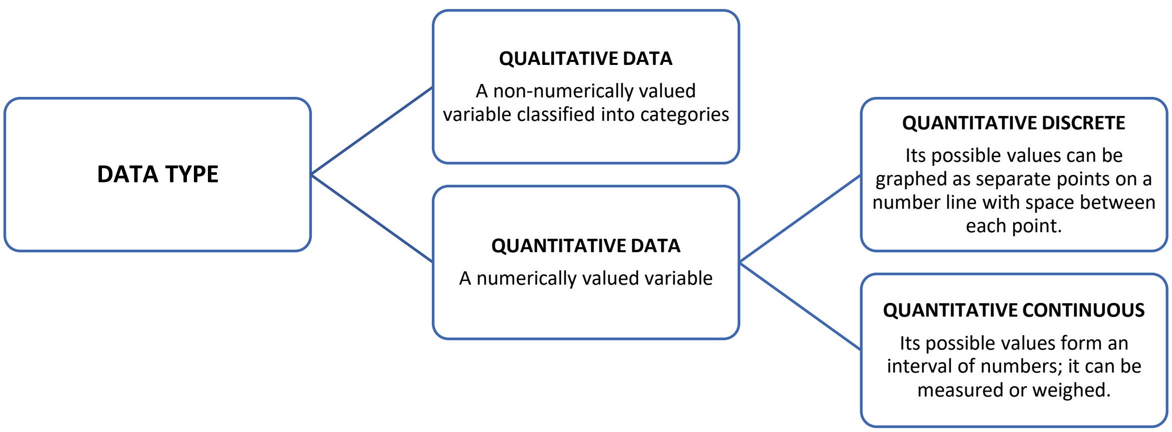


Figure 1 Data type classification

Example 1. Eye color

An eye clinic collected the eye color of their patients. Eye color is classified as qualitative data because eye color consists of values that are categories (such as brown, blue, green, hazel, and black).

Example 2. People's Height

The variable “heights of people” takes on an infinite number of possible values. It is classified as quantitative continuous data. Even though a person's height might be 5'8", it is really 5.68241231... feet. It is impossible to measure a length exactly!

Example 3. Household Size

The variable number of people in a household takes on a finite number of values. These possible values are 1, 2, 3, ... which can be plotted on a number line as points with space between each point. Thus, number of people in a household is classified as quantitative discrete data.

Example 4.

The table below contains data organized by variables about a small group of CCBC students. The variables are: Student Name, Gender, No. of classes completed, GPA, Student ID. We would like to identify the data type (variable) presented in each column as either qualitative, quantitative discrete, or quantitative continuous.

Name	**Gender**	**No. of classes completed**	**GPA**	**Student ID**
Tim	Male	4	3.25	800567230
Joy	Female	9	3.75	800213457
Brad	Male	6	4.00	800120985
Alice	Female	5	3.90	800900521
Michael	Male	10	3.81	800749541

Students’ name, as well as students' gender, takes on values that are categories. The data in both of these cases are classified as qualitative.

The variable “number of classes completed” per student is finite. Its possible values are 0, 1, 2, 3, ...; these can be plotted on a number line as points with space between each point. Thus, the data in these cases are classified as quantitative discrete.

GPA can take an infinite number of possible values in the interval 0.0 to 4.0. So, these data are classified as quantitative continuous.

Although the variable Student ID consists of numbers, the numbers are used as labels Thus these data are classified as qualitative.

5.1 Practice

You plan to purchase a car and you have your mind set on a few car characteristics (variables). Determine if these variables are either *quantitative discrete, quantitative continuous,* or *qualitative.*

1) Your car budget limits
2) Car Make
3) Number of cylinders
4) Car Model
5) The braking distance measured on a scale from 100 ft. to 200 ft

It is Orioles opening season game day. Determine if the following variables are either *quantitative discrete, quantitative continuous,* or *qualitative*.

6) The high temperature of the day.
7) Birth country of the players.
8) Number of home runs during the game.
9) Shirt numbers on athletes uniforms.

10) **The table below provides information of some specifications for five smartphones. Identify the type of data presented in each column as either *qualitative, quantitative discrete,* or *quantitative continuous*.**

Rank	Smartphone	Weight (oz)	Touch ID	Battery (talk time in min.)
1	Apple iPhone 6S	5.04	Yes	840
2	Apple iPhone 6S Plus	6.77	Yes	1440
3	Samsung Gallaxy S7	5.37	Yes	1680
4	Sony Xperia Z	5.15	No	840
5	LG G5	5.57	No	1320

11) **The table below provides information of some specifications for five breakfast cereals. Identify the type of data presented in each column as either *qualitative, quantitative discrete*, or *quantitative continuous*.**

Cereal Name	Cereal Brand	Calories (per cup)	Sugar (per cup)	Gluten Free
Special K	Kellogg	110	9g	No
Cheerios	General Mills	100	1g	Yes
Oats and Honey	Cascadian Farm	260	14g	No
Corn Flakes	Millville	100	2g	No
Rice Chex	General Mills	100	2g	Yes

12) Below is a small survey about driving history. Identify the type of data being collected with each survey question as either *qualitative*, *quantitative discrete*, or *quantitative continuous*.

a. How old are you?
b. What is your gender?
c. Have you taken a driver's education course?
d. What is the fastest speed you have driven an automobile?
e. How many speeding tickets have your received in your lifetime?

13) You have decided to collect some data on your Netflix account. Identify the type of data you are collecting as either *qualitative*, *quantitative discrete*, or *quantitative continuous*.

a. The genre of the film.
b. The length (in minutes) of the film.
c. The number of family members that have access to the account.

5.1 Answers

1) Quantitative continuous
2) Qualitative
3) Quantitative discrete
4) Qualitative
5) Quantitative continuous
6) Quantitative continuous
7) Qualitative
8) Quantitative discrete
9) Qualitative
10) Qualitative data: Rank, Smartphone name, Touch ID.
    Quantitative continuous: Weight, Battery talk time
11) Qualitative data: Cereal Name, Cereal Brand, Gluten Free
    Quantitative continuous: Calories (per cup), Sugar (per cup)

12)

   a. Quantitative continuous
   b. Qualitative
   c. Qualitative
   d. Quantitative continuous
   e. Quantitative discrete

13)

   a. Qualitative
   b. Quantitative continuous
   c. Quantitative discrete

Section 5.2: Organizing and Graphing Categorical Data

Objective: Create a frequency table.

Data is being collected all the time by businesses, governments, and researchers. The data can range from small to quite large. We need to be able to better understand the nature of the data. Organizing it helps! In this section, we will organize qualitative data. Because qualitative data is classified into categories, we will refer to this data as categorical data.

Definition. *A **frequency table** shows how data are divided among several categories (or classes) by listing the categories along with the number (frequency) of data values in each of them.*

Example. Coffee Shop
A local coffee shop keeps a list of types of drinks that their customers order each hour. Below is the data from 50 drinks sold an hour before closing on a recent Tuesday.

Coffee	Espresso	Coffee	Tea	Espresso
Tea	Coffee	Tea	Coffee	Espresso
Espresso	Tea	Espresso	Espresso	Espresso
Espresso	Espresso	Coffee	Espresso	Tea
Coffee	Soda	Espresso	Coffee	Coffee
Espresso	Tea	Espresso	Soda	Tea
Coffee	Espresso	Coffee	Tea	Espresso
Coffee	Soda	Coffee	Coffee	Espresso
Soda	Espresso	Tea	Espresso	Coffee
Coffee	Espresso	Coffee	Espresso	Tea

A frequency table provides a useful way to organize this data. There are four different categories of drinks they sell (Coffee, Espresso, Soda, and Tea). They are in the first column of the table. The second column contains the counts of each type sold that hour.

Drink	Frequency
Coffee	16
Espresso	20
Soda	4
Tea	10

Objective: Compute relative frequencies.

An additional column can be added to the table to give us a better understanding of the data. This column is called a relative frequency. It can be found by dividing the frequency count for a category by the sum of all frequency counts.

$$\textbf{Relative frequency for a category} = \frac{\textbf{frequency of a category}}{\textbf{sum of all frequencies}}$$

Drink	**Frequency**	**Relative Frequency**
Coffee	16	$\frac{16}{50} = 0.32$
Espresso	20	$\frac{20}{50} = 0.40$
Soda	4	$\frac{4}{50} = 0.08$
Tea	10	$\frac{10}{50} = 0.20$

If we turn the relative frequency into a percent we get a better understanding of the sales that hour.

$$\textbf{Percentage for a category} = \frac{\textbf{frequency of a category}}{\textbf{sum of all frequencies}} \times \textbf{100\%}$$

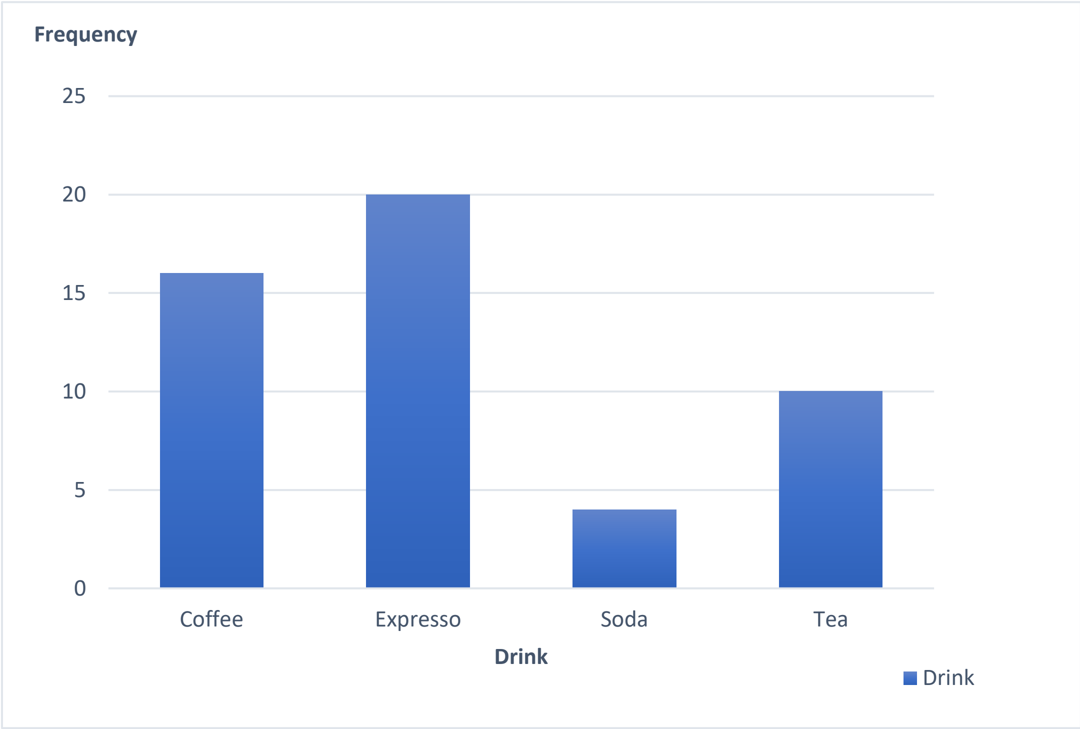
Drink	**Frequency**	**Relative Frequency**	**Percentage**
Coffee	16	$\frac{16}{50} = 0.32$	$0.32 \times 100\% = 32\%$
Espresso	20	$\frac{20}{50} = 0.40$	$0.40 \times 100\% = 40\%$
Soda	4	$\frac{4}{50} = 0.08$	$0.08 \times 100\% = 8\%$
Tea	10	$\frac{10}{50} = 0.20$	$0.20 \times 100\% = 20\%$

Initially it was easy to see that Espresso is the most popular drink, but looking at the relative frequency or percentage gives us a better idea of how it relates to the other drink choices. The manager can use information such as this to better stock the shop.

Objective: Create a bar graph.

Nothing makes a report look better than a nice graph. In addition to creating frequency tables, an analyst might want to create a graph of categorical data. There are many different types of graphs. **Bar graphs** are probably the most commonly used graphs and they are used to compare things between different groups.

Here is a bar chart of our coffee shop data. The categories are along the horizontal axis and the frequency counts correspond to the height of the bars.

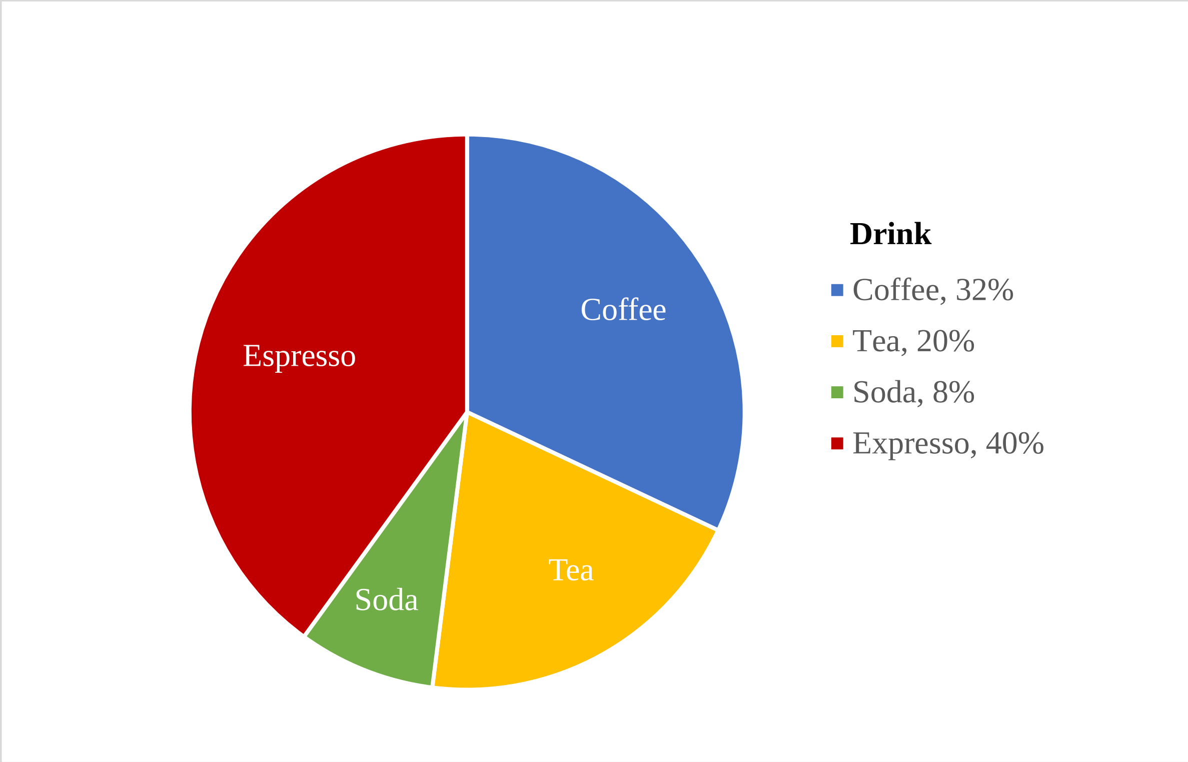


Rules when constructing a bar graph

1. The height of each bar represents the frequency or relative frequency for that category.
2. The bars should be of the same "width."
3. The bars should not overlap.
4. Each piece of data should belong to only one category.

Objective: Create a pie chart.

Like bar graphs, pie charts are very common to graph categorical data. **Pie charts** show how the size of the category relates to the whole group. Pie charts are great for showing percentages. Below is the coffee shop pie chart. Notice how the percentages correspond to the size of the pie pieces.



Rules when constructing a pie chart

1. Always include the relative frequency or percentage.
2. Include labels, either as a legend or directly on pie.

5.2 Practice

1. Twenty-four students answered a survey about pet preferences. Their responses are below.

Cat	Guinea pigs	Guinea pigs	Cat	Rabbit	Dog	Guinea pigs	Dog
Dog	Cat	Dog	Dog	Guinea pigs	Dog	Cat	Rabbit
Dog	Rabbit	Cat	Guinea pigs	Dog	Cat	Rabbit	Dog

   a) Construct a frequency table for this data.
   b) Draw a bar graph.
   c) How many students participated in this survey?
   d) What percent of students like dogs?

2. In a software engineering class, the professor asked his students to name their favorite programming language. Their replies are listed in the table below.

Java	Lisp	Perl	Java	Perl	Perl	Perl	C++	Perl	Java
Perl	Java	Java	Perl	Java	Lisp	Java	Perl	Java	Lisp
Perl	C++	C++	Perl	C++	Perl	Java	C++	Perl	C++

   a) Construct a frequency table for this data.
   b) Draw a bar graph.
   c) How many students participated in this survey?
   d) What percent of students like C++?

3. The following frequency table represents the number of new HIV/AIDS cases in the US in 2008 according to race/ethnicity. What percent of the new cases were Hispanic/Latino?

Race/Ethnicity	**Number of HIV/AIDS Cases**
American Indian/Alaskan Native	228
Asian	451
Black/African American	21,443
Hispanic/Latino	7,461
Native Hawaiian/other Pacific Islander	47
White	12,534

4. A school district performed a study to find the main causes leading to its students dropping out of school. Fifty cases were analyzed and a primary cause was assigned to each case. The results for the fifty cases are listed below. What percent of students drop out due to family problems?

Causes to drop out of school	**Frequency**
Unexcused absences	12
Illness	16
Family problems	14
Other causes	8

5. Relative frequencies allow us to compare groups. Here is the 2008 new HIV/AIDS cases in the US separated by sex.

Males	
Race/Ethnicity	**Frequency**
Black	14,247
White	10,563
Hispanic	5,906
Other	565

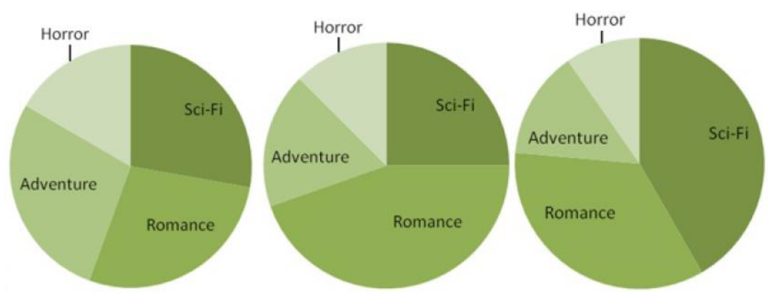
Females	
Race/Ethnicity	**Frequency**
Black	7,196
White	1,971
Hispanic	1,555
Other	161

a) Compute the relative frequencies for each sex.
b) Write a few sentences explaining the trend of new cases in 2008 using what you learned in part a.

6. The table below represents 360 books grouped by their category:

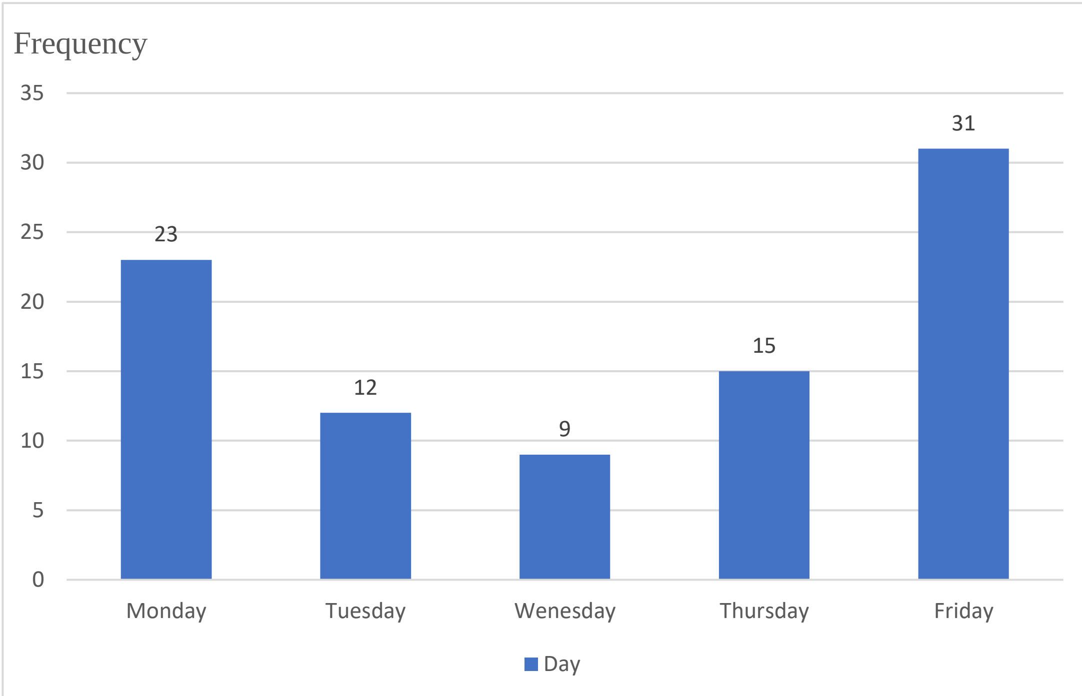
Book Category	**Frequency**
Science Fiction	150
Romance	125
Adventure	50
Horror	35

Which pie chart best represents this table?

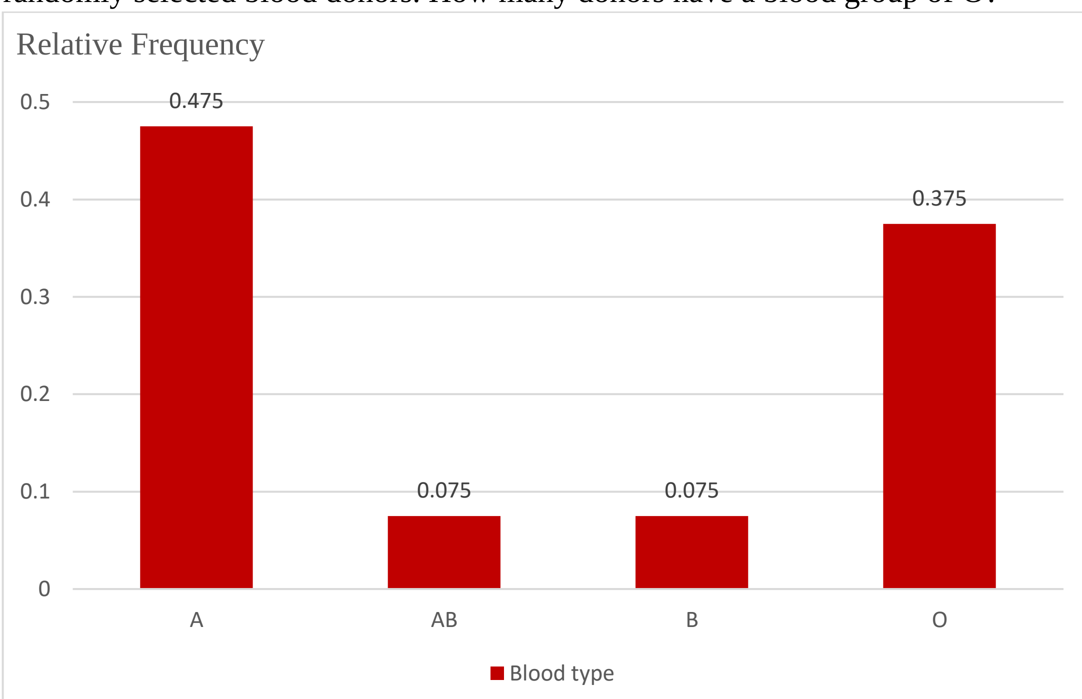


(a) (b) (c)

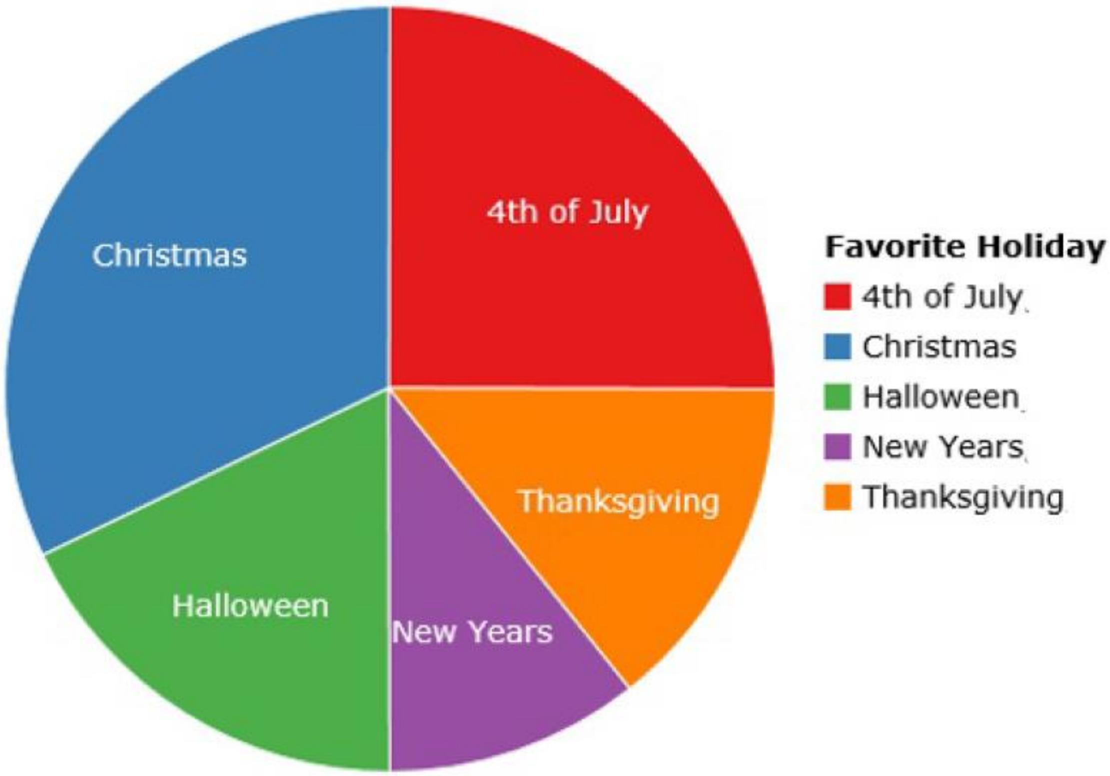
7. The bar chart below describes the day of the week workers called in sick for workers at a company. What is the relative frequency for Monday?



8. The bar chart below show the blood groups of O, A, B, and AB of a group of forty randomly selected blood donors. How many donors have a blood group of O?



9. The pie chart below shows the student responses to a survey asking them about their favorite holiday. Use the graph to find the percent of students who answered “4th of July”.

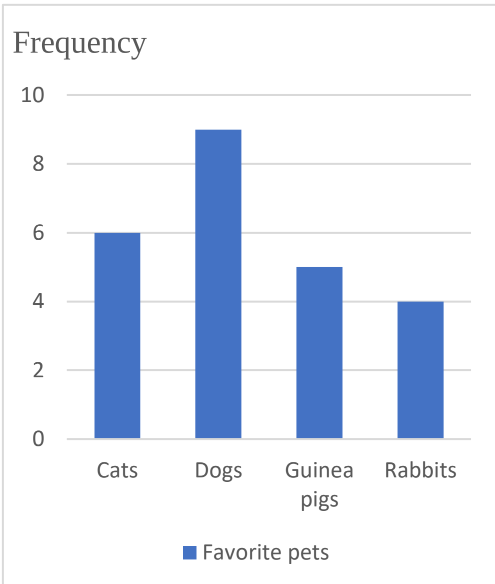


5.2 Answers

1. a)

Favorite pets	Frequency
Rabbits	4
Cats	6
Dogs	9
Guinea pigs	5
Total	24

b)

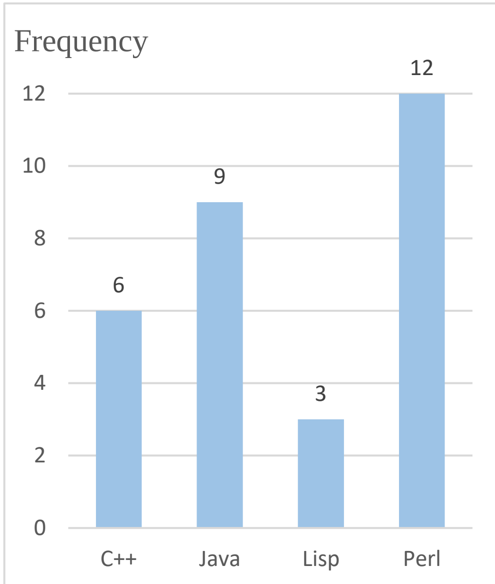


c) 24 students participated in the survey
d) 37.5% of students like dogs

2. a)

Programming language	**Frequency**
C++	6
Java	9
Lisp	3
Perl	12
Total	30

b)



c) 30 students participated in the survey

d) 20% of students like C++

3. 17.7%
4. 28%
5. a)

Males	
Race/Ethnicity	**Relative Frequency**
Black	0.455
White	0.338
Hispanic	0.189
Other	0.018

Females	
Race/Ethnicity	**Relative Frequency**
Black	0.661
White	0.181
Hispanic	0.143
Other	0.015

b) When you are comparing new cases of HIV among men and women their relative frequencies for race are nothing alike. With both males and females, the majority of the cases are with Blacks and Whites. Females have two thirds of new cases just among Blacks. The epidemic of HIV is very different according to race for men and women.

6. Pie chart (c)
7. 0.2556
8. 15
9. 25%

Section 5.3: Organizing and Graphing Quantitative Data

Objective: Organize quantitative data into frequency tables

An easy way to compile quantitative data would be to make a frequency or relative frequency table as we did with qualitative data.

Commute time to school

The data in the table below are the commute time to school (in minutes) for a group of students attending a particular mathematics course. We would like to create a frequency table.

10	15	17	20	25	20	3	30	17	15	5	15
60	8	25	15	25	22	38	10	14	30	30	18

Since there are no natural categories for this data we must create what are called classes. Classes divide the number line into smaller pieces.

First let's find the minimum commute time (3 minutes). Next, find the maximum commute time (60 minutes). We can start our first class at 3 minutes or back up a few minutes. Let's start the first class at 1 minute and use a class width of 10 minutes. The first class is "1 – 10". The 1 is considered the lower limit of the class and the 10 is considered the upper limit of the class.

1 – 10

Lower class limit   Upper class limit

Next, we will determine the lower limits of the other classes. Add the class width of 10 to our first lower limit. This will give us 11. If we continue to add 10 the remaining lower limits will be 21, 31, 41, and 51. The upper limits are determined by filling in the numbers that approach the next class' lower limit but do not equal it. For example in the second row we chose 20 because it is the closest whole number less than 21.

The difference is 10

Commute Time
1 – 10
11 – 20
21 – 30
31 – 40
41 – 50
51 – 60

The class width of 10 is evident by looking at the differences in consecutive lower class limits. This is also the case with consecutive upper class limits. They are all 10. Once we made all the classes and made sure that the minimum and maximum can be placed in the table, we see that there are 6 classes. Now we must determine how many commute times fall into each of the classes. There are several strategies to tallying up the counts. You can mark them off as you go, or possibly put unique symbols or marks next to the ones that are in the same classes. This will help you avoid classifying a value twice or forgetting a data value altogether. Once all the symbols are there you can count them.

10 *	15🖑	17🖑	20🖑	25❖	20🖑	3 *	30❖	17🖑	15🖑	5 *	12🖑
60■	8 *	25❖	15🖑	25❖	22❖	38⌘	10 *	14🖑	30❖	30❖	18🖑

	Commute Time	Frequency
*	1 – 10	5
🖑	11 – 20	10
❖	21 – 30	7
⌘	31 – 40	1
☒	41 – 50	0
■	51 – 60	1

Vocabulary:

Classes = range of data values used to group the data.
Lower class limit = the smallest value that goes in a class.
Upper class limit = the largest value that goes in a class.
Class width = the difference between two consecutive lower class limits.

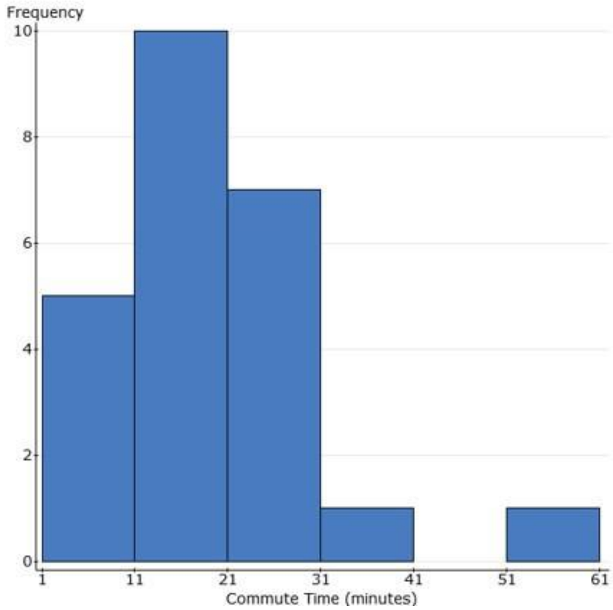
Objective: Create a histogram for quantitative data

We can take our frequency table and create a graph of the information. This graph is called a histogram. A histogram is like a bar graph. The classes are along the horizontal axis and the frequencies are demonstrated with the vertical axis. The bars need to touch in a histogram because we want to imply that the classes are adjacent and represent a continuum on the number line.

Steps to sketch a histogram:

1) Draw the horizontal axis with the lower class limits equally spaced along it.
2) Draw the vertical axis with the frequencies equally spaced along it.
3) Create the bars (rectangles)

The histogram for the "Commute time to school" is shown below.



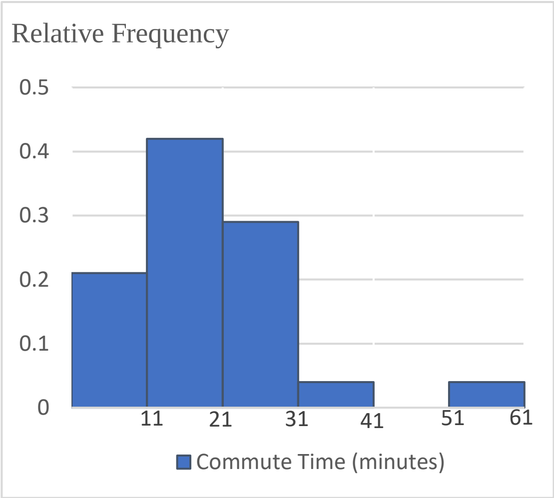
You can see the height of the bars are exactly as our frequency table (5, 10, 7, 1, 0, and 1). The lower class limits are along the horizontal axis. The first bar gives us the commute times that were in the class 1 – 10. A histogram allows us to visualize the data and see how the data is spread out or possibly similar to one another. Notice there is not a bar for 41 – 50. This is because the frequency was zero for that class. Frequencies of zero are the only reasons there are gaps in histograms.

Objective: Create Relative Frequencies

We can also add a column to the frequency table to represent relative frequency. This is similar to what was done for frequency tables for categorical data.

$$\text{Relative frequency for a class } = \frac{\text{frequency in that class}}{\text{sum of all frequencies}}$$

Commute Time	Frequency	Relative Frequency
1 – 10	5	$\frac{5}{24} = 0.21$
11 – 20	10	$\frac{10}{24} = 0.42$
21 – 30	7	$\frac{7}{24} = 0.29$
31 – 40	1	$\frac{1}{24} = 0.04$
41 – 50	0	0
51 – 60	1	$\frac{1}{24} = 0.04$

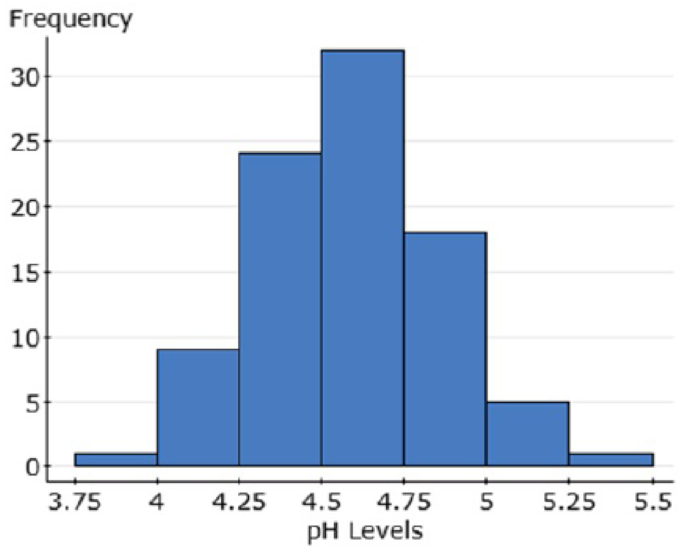


The relative frequencies, like the histogram, allow us to see that the majority of the class had a commute time between 11 and 30 minutes.

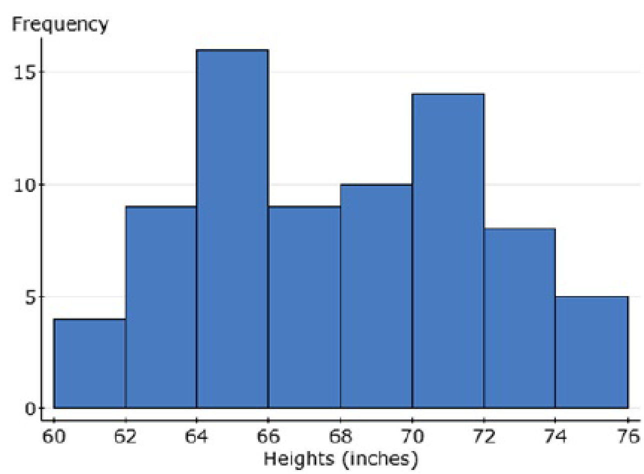
Objective: Discuss the shape of histograms

Histograms are valuable tools to display data. There are features that we tend to describe with words. For example, it is helpful to mention how many peaks (humps) are present in the graph. Does the histogram have a single, central peak or several separated peaks? A histogram with one main peak is dubbed **uni**modal; histograms with two peaks are **bi**modal; histograms with three or more peaks are called **multi**modal.

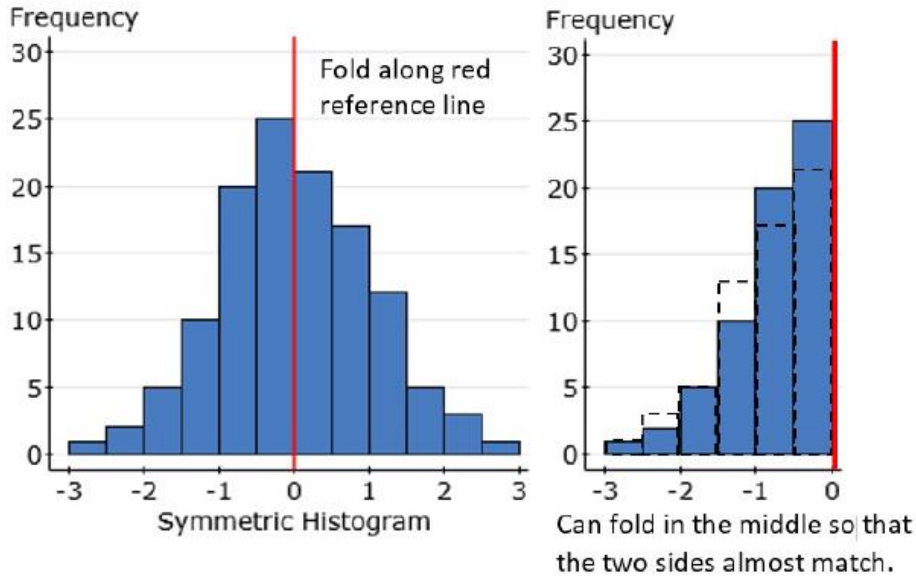
This is an example of a **unimodal** histogram. This histogram displays 90 different rainfalls at a national park. They measured the pH level of each rainfall.



This is an example of a bimodal histogram. These are the heights of 75 singers in a chorus.

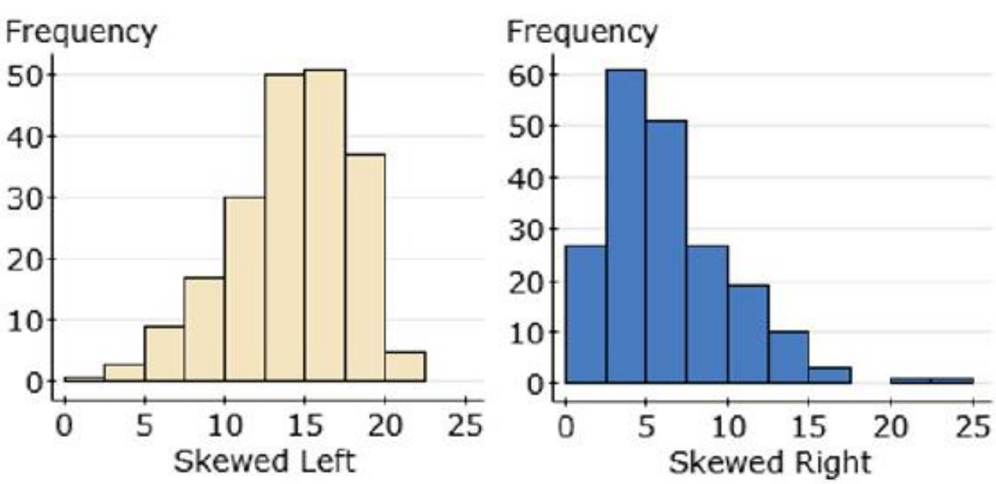


Another way we describe histograms is to discuss their symmetry (or lack there of). If you can fold the histogram along a vertical line through the middle and have the edges match pretty closely, the histogram is considered symmetric.



The (usually) thinner ends of a distribution are called the tails. If one tail stretches out farther than the other, the histogram is said to be skewed to the side of the longer tail.

Below the histogram on the left is said to be skewed left, while the histogram on the right is said to be skewed right.



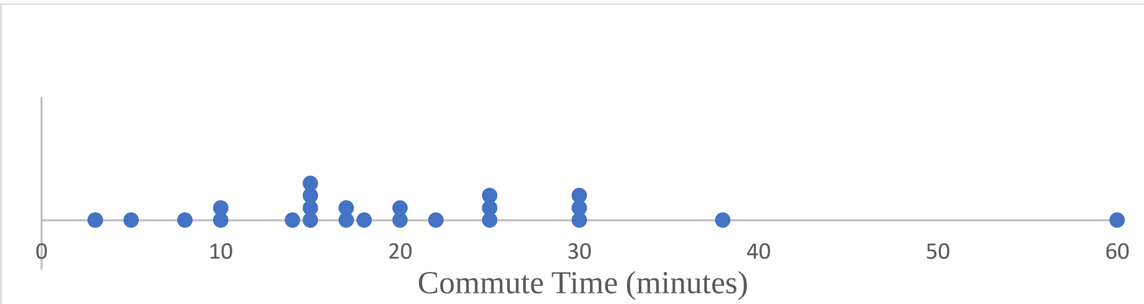
Objective: Create dotplots for quantitative data

Definition. *A* **dotplot** *is a visual representation of quantitative data and provides a graphical display of the data distribution*.

<u>Steps to construct a dot plot:</u>

1) Begin by drawing a number line that reflects the range of values.
2) Plot each data point by placing a dot over the appropriate value. For any repeated value stack the points.

The dotplot for the "Commute time to school" is shown below.



Using the dotplot we can answer many questions about our data.

What observation occurred the most?	We have four stacked dots above 15, so we conclude that 15 minutes is the observation that occurred the most.
How many students were surveyed?	Each point represents a reported commute time. By counting all the points on the graph we get the total number of students surveyed. We have 24 dots so the answer is 24 students.
What proportion of students commute to school in 20 minutes or less?	Count the points on the graph that are stacked at the value 20 or below it. 15 students commute in 20 minutes or less. Divide 15 by 24 (total students) to get the proportion. The answer is 15/24=0.625

Objective: Create stem-and-leaf plots for quantitative data

Definition. ***A stem-and-leaf plot*** *is another way to arrange quantitative data. The plot separates each value into two parts: the stem (such as everything of the leftmost digits) and the leaf (such as the rightmost digit).*

<u>Steps to construct a stem-and-leaf plot:</u>

1) Draw a vertical bar

2) Identify the stems and list them down on the left side of the vertical bar
3) List to the right of the vertical bar the leaves corresponding to their stems

Consider the Commute to school data

10	15	17	20	25	20	3	30	17	15	5	15
60	8	25	15	25	22	38	10	14	30	30	18

For this data, the stems are the tens, namely, 0, 1, 2, 3, 4, 5, 6 and the leaves are all the ones. For example, the data value 10 is separated into its stem of 1 and the leaf of 0. The data value 15 is separated into its stem of 1 and the leaf of 5. All the values are separated in the same way and arranged as shown below.

0	358
1	0045555778
2	002555
3	0008
4	
5	
6	0

2 | 5 means 25

Note: The leaves are arranged in increasing order.

5.3 Practice

1) The frequency distribution below summarizes employee years of service for a particular corporation.

Years of service	Frequency
1 – 5	4
6 – 10	15
11 – 15	20
16 – 20	9
21 – 25	5
26 – 30	2

   a) How many employees participated in the survey?
   b) What is the class width?
   c) Identify the lower class limits.
   d) Identify the upper class limits.

2) A business magazine was conducting a study into the amount of travel required for mid-level managers across the US. They surveyed a group of managers and asked them the number of days they spent traveling each year. The frequency distribution below summarizes the results.

Days Traveling	Frequency
0 – 6	15
7 – 13	21
14 – 20	27
21 – 27	9
28 – 34	2
35 – 41	1

   a) How many managers participated in the survey?
   b) What is the class width?
   c) Identify the lower class limits.
   d) Identify the upper class limits.

3) A teacher at your college selected a group of students from all of his classes and asked them the number of credits each took that semester.

9	6	10	3	9	7	10	9	12	9	6	9
12	9	7	10	9	6	9	7	7	14	9	8

a) Create a frequency and a relative frequency table to display the data. Let your first lower limit be 3 and the class width be 3 credits.
b) Create a frequency histogram.
c) Create a relative frequency histogram.
d) How does the frequency histogram and relative frequency histogram compare?

4) Twenty-four students were asked the number of hours they sleep each night. The results of the survey are listed below.

7	9	5	8	8	11	12	7	10	7	8	9
8	6	4	3	9	10	7	6	12	5	6	11

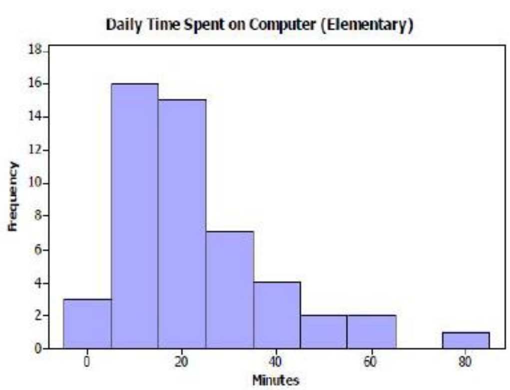
a) Create a frequency and a relative frequency table to display the data. Let your first lower limit be 3 and the class width be 2 hours.
b) Create a frequency histogram.
c) Create a relative frequency histogram.
d) How does the frequency histogram and relative frequency histogram compare?

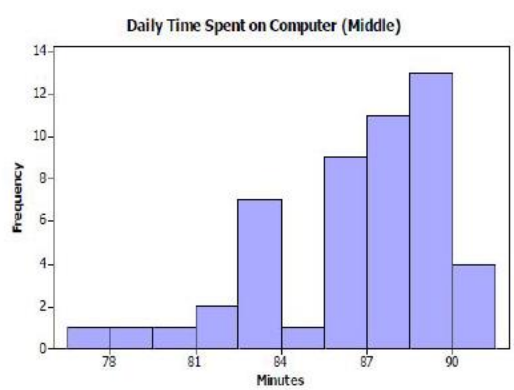
5) In a survey, 26 voters were asked their ages. The results are shown below.

43	56	28	63	67	66	52	48	37	56	40	60	62
66	45	21	35	49	32	56	61	53	69	31	48	59

a) Create a frequency table for the ages. Let your first lower limit be 19 and the class width be 10.
b) Create a frequency histogram.

6) The following histograms describe the number of minutes per day that a group of 50 elementary and 50 middle school students spent on a computer.

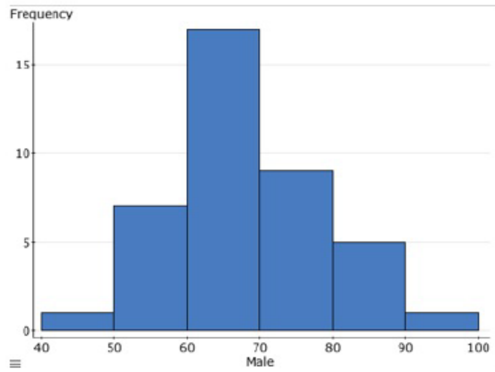


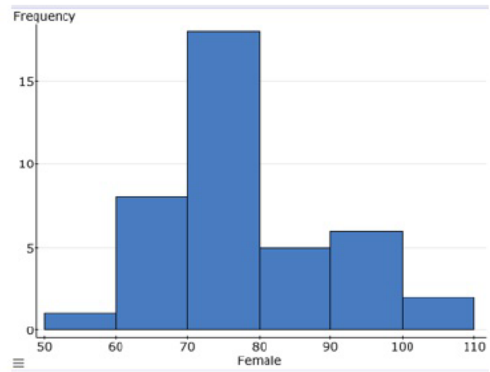


a) Which histogram appears to be skewed to the right?

b) Which histogram appears to be skewed to the left?

7) The following histograms represent the pulse rate (beats per minute) for a group of 40 males and 40 females.





a) Describe the shape of the two histograms.
b) Compare the histogram of pulse rate of males with females. Is there a notable difference between pulse rates of females and males?

8) The stem and leaf plot below shows the number of laps run by each participant in a marathon. Refer to the stem and leaf plot to answer the following questions.

1	6					
2						
3	0	0	0	5	5	
4	3	3	8	8	8	8
5	0	1	1	1	1	
6	2					

a) Construct a dotplot.
b) How many participants ran less than 48 laps?
c) How many participants ran at least 50 laps?
d) What observation occured the most?

9) The stem and leaf plot below shows the ages of a group of patients who had strokes caused by stress. Refer to the stem and leaf plot to answer the following questions.

2	9					
3	0	6	6			
4	1	4	7	7		
5	3	3	8	8	8	8
6	0	0	1	1		
7	2	4	4	6	9	

a) Construct a dotplot.
b) How many patients were at most 36 years old?
c) How many patients were 72 or older?
d) What observation occurred the most?

5.3 Answers

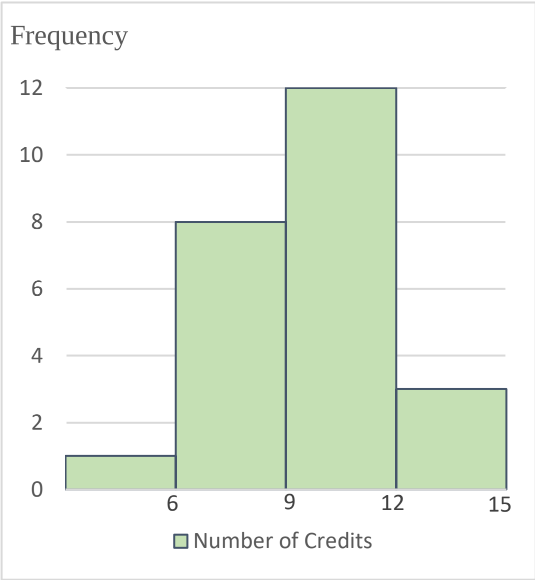
1.
   a) 55 employees
   b) Class width: 5
   c) Lower class limits: 1, 6, 11, 16, 21, 26
   d) Upper class limits: 5, 10, 15, 20, 25, 30

2.
   a) 75 managers
   b) Class width: 7
   c) Lower class limits: 0, 7, 14, 21, 28, 35
   d) Upper class limits: 6, 13, 20, 27, 34, 41

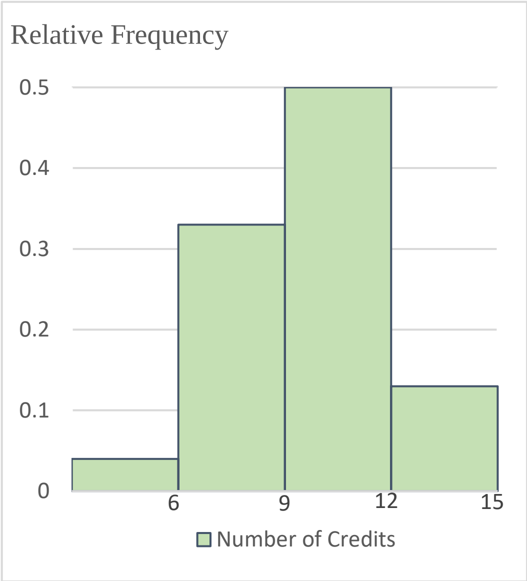
3.
   a)

Number of credits	Frequency	Relative Frequency
3 – 5	1	0.04
6 – 8	8	0.33
9 – 11	12	0.50
12 – 14	3	0.13

b)



c)



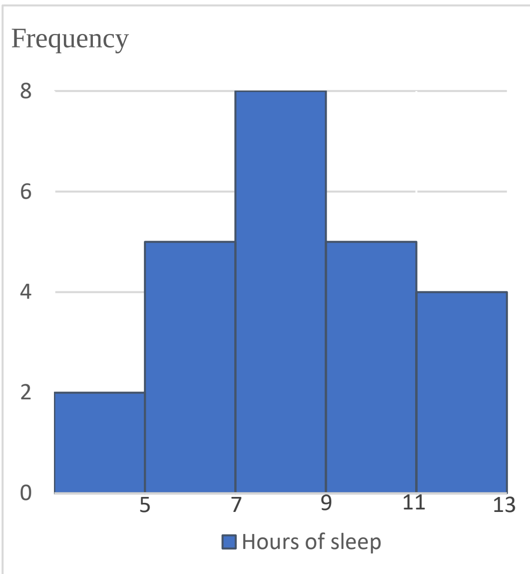
d) The frequency histogram and relative frequency histogram have the same shape because frequencies and relative frequencies are proportional. The only difference is on the units of the $y-$axis.

4.

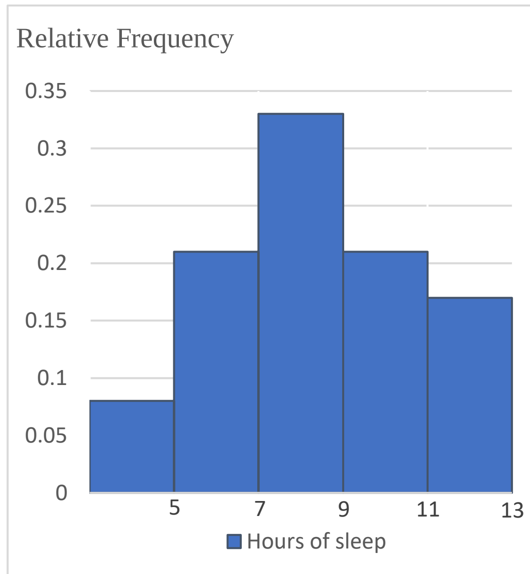
a)

Hours of Sleep	Frequency	Relative Frequency
3 – 4	2	0.08
5 – 6	5	0.21
7 – 8	8	0.33
9 – 10	5	0.21
11 – 12	4	0.17

b)



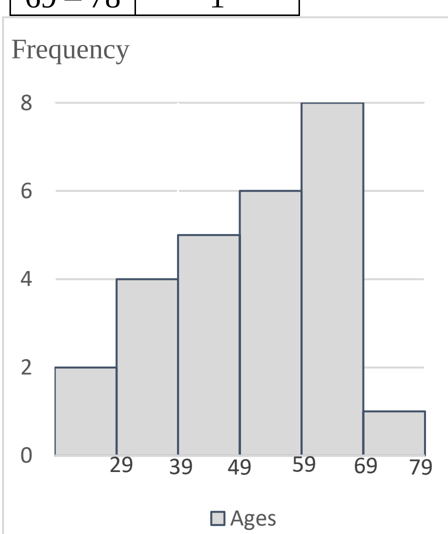
c)



5. a)

Age	Frequency
19 – 28	2
29 – 38	4
39 – 48	5
49 – 58	6
59 – 68	8
69 – 78	1

b)



6.

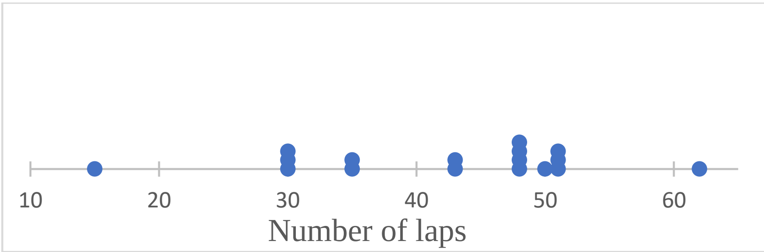
a) Elementary school students' histogram appears to be skewed to the right.
b) Middle school students' histogram appears to be skewed to the left.

7.

a) The histogram of pulse rate of males appears to be unimodal and approximately symmetric. The histogram of pulse rate of females appears to be bimodal and skewed to the right.

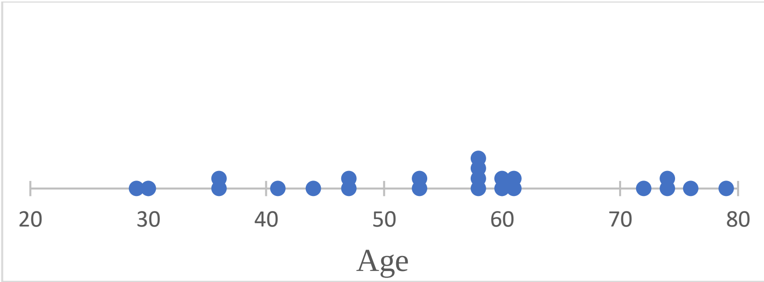
b) The pulse rates of males appear to be generally lower than the pulse rates of females.

8.

a) 

b) 8 participants ran less than 48 laps
c) 5 participants ran at least 50 laps
d) 48 occured the most

9.

a) 

b) 4 patients were at most 36 years old
c) 5 patients were 72 and older
d) 58 occured the most

Chapter 6: Descriptive Measures

Section 6.1 Measures of Center

Objective: Compute a mean

In this lesson we are going to continue summarizing data. Instead of using tables and graphs we are going to make some numerical calculations that will aid in our understanding of data.

Definition. *The **mean** of a set of data is the measure of center found by adding the data values and dividing the total by the number of data values.*

Mean:

$$\mu = \frac{\text{sum of all the data values}}{\text{number of data values}}$$

Example 1. Shoplifters

The following data represents the number of shoplifters per week at a large electronics store for thirteen weeks. Calculate the mean number of shoplifters/week.

1	7	0	2	4	5	2	5	4	1	3	1	6

$$\mu = \frac{1+7+0+2+4+5+2+5+4+1+3+1+6}{13}$$

$$\mu = \frac{41}{13}$$

$$\mu = 3.2\ shoplifters\ per\ week$$

The mean is a one number summary that could describe the number of shoplifters for a typical week at the store.

Objective: Compute a median

Definition. *The **median** of a data set is the measure of center that is the middle value when the data values are sorted from smallest to largest. This measure splits the data in 2 equal parts. Half of the data values are below the median and half of the data values are above the median.*

Formula to help us locate the median:

$$\text{Position of median } = \frac{n+1}{2}$$

***Note**: This is NOT the median. It is just its <u>location</u> in our sorted data.*

Example 2. Shoplifters

We will revisit the shoplifter example. This is the same data except that it is sorted from smallest to largest.

1	7	0	2	4	5	2	5	4	1	3	1	6

$$\text{Position of median } = \frac{13+1}{2} = 7$$

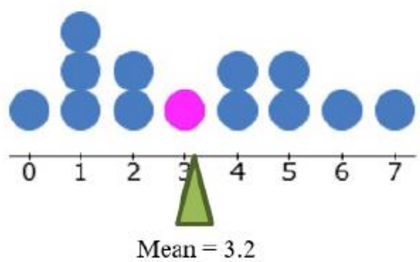
Our median is the 7th value in our sorted data!

0	1	1	1	2	2	3	4	4	5	5	6	7
1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th

Median = 3 shoplifters/week

The median <u>does</u> slice the data in *half*. Six values are below it and above it.

Another way to think of the **mean** is that it is the balance point of the data. The fulcrum (triangle under dotplot) shows where this dot plot would balance. Notice that its location is the same as our mean. The purple dot is our median. It is a colored dot because it is an actual observation in the data.



Example 3. Shoplifters again

Suppose a 14th week was added. Recalculate the mean and median.

Mean:

1	7	0	2	4	5	2	5	4	1	3	1	6	**30**

$$\mu = \frac{1+7+0+2+4+5+3+5+4+1+3+1+6+30}{14}$$

$$\mu = \frac{71}{14}$$

$$\mu = 5.07 \; shoplifters/week$$

Median:

$$\text{Position of median } = \frac{14+1}{2} = 7.5$$

This tells us that our median is between the 7th and 8th values in our sorted data! When there is an even number of data values in a data set the median will always be between two data values.

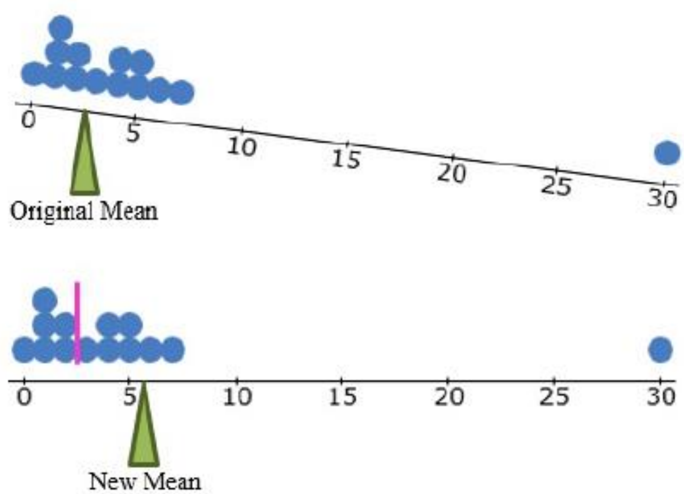
0	1	1	1	2	2	3	4	4	5	5	6	7	30
1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th	14th

We will need to determine what number is exactly between the 7th and 8th data values. The midpoint formula will help us determine this.

$$\text{midpoint} = \frac{3+4}{2}$$
$$\text{midpoint} = \frac{7}{2}$$

Median = 3.5 shoplifters/week

If we take a look at the dot plot again you can see that the 30 is very different from the other weeks and it creates an imbalance.

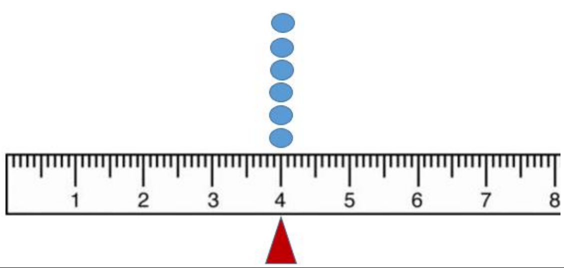
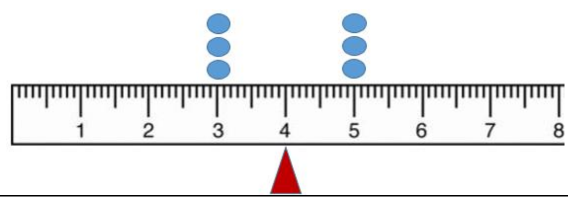


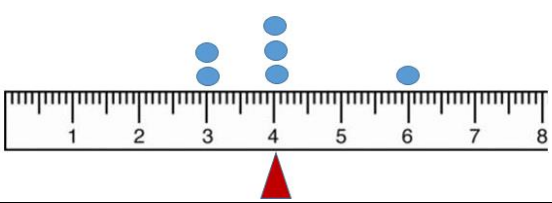
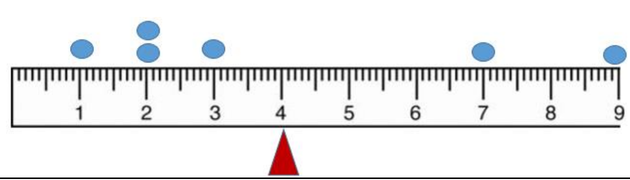
The new mean with the 30 included has shifted to the right to restore the balance. The median is again in purple. It is a line because it is not an actual value in our data this time. It still divides the data into two equal parts. Did the median react much to the value of 30? No, medians are not affected by extreme values in data sets. Such extreme values are called outliers.

The mean and median measure the center of data in different ways. The mean is the balance point. The median is the middle number (half the data values are less than the median and half are greater).

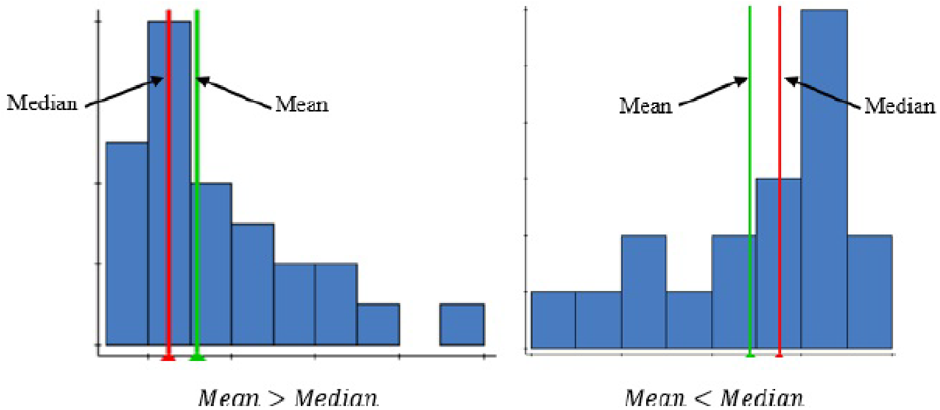
Example 3. Balance point is the mean.

Not all data are created equal. Imagine a dataset that has six values and a mean of 4. Here are just a few below. The red triangle represents the mean as well as the balance point.

Data 1: 4, 4, 4, 4, 4, 4	**Data 2:** 3, 3, 3, 5, 5, 5
$\mu = \frac{4+4+4+4+4+4}{6}$ $\mu = \frac{24}{6}$ $\mu = 4$	$\mu = \frac{3+3+3+5+5+5}{6}$ $\mu = \frac{24}{6}$ $\mu = 4$
	

Data 3: 3, 3, 4, 4, 4, 6	**Data 4:** 1, 2, 2, 3, 7, 9
$\mu = \frac{3+3+4+4+4+6}{6}$ $\mu = \frac{24}{6}$ $\mu = 4$	$\mu = \frac{1+2+2+3+7+9}{6}$ $\mu = \frac{24}{6}$ $\mu = 4$
	

When data is skewed to the right, the mean is generally larger than the median. When the data is skewed to the left the mean is generally smaller than the median.

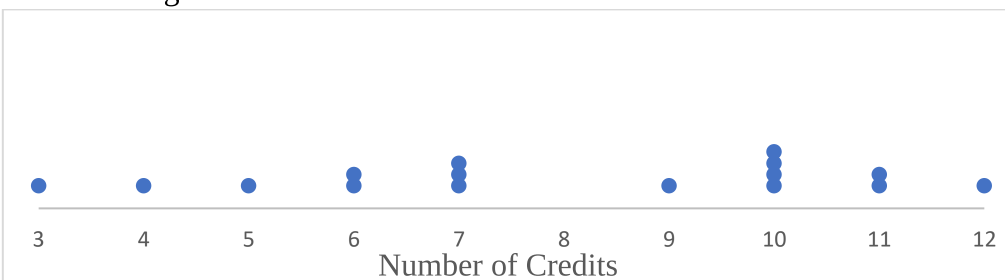


6.1 Practice

1. The following data represent the number of pop-up advertisements received by 11 families during the past month. Calculate the mean and median number of advertisements received by each family during the month.

43	37	35	30	41	23	33	31	16	21	39

2. The following data represents the number credits of students in a College Algebra class were taking in fall 2014.



   a. How many students were in the class?
   b. What is the mean number of credits?
   c. What is the median number of credits?

3. The following are daily dollar amounts collected at a neighborhood lemonade stand:

19	55	25	37	32	28	22	23	29	34	39	31	26	17

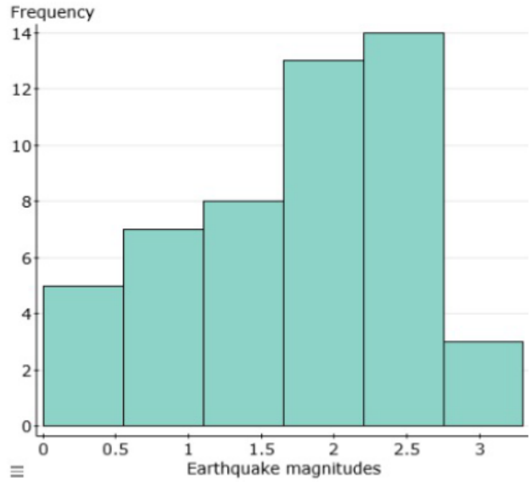
   a. Compute the mean.
   b. Compute the median.

4. The times spent waiting in line (in minutes) for 21 randomly selected customers during the lunch rush hour at a local fast food restaurant are below. Use these data to answer the questions below.

2.0	1.8	4.0	1.5	1.0	3.4	2.3	3.2	4.5	3.2	2.5
3.1	2.5	3.4	5.1	3.5	3.2	3.5	2.9	4.2	2.7	

   a. Compute the mean
   b. Compute the median.

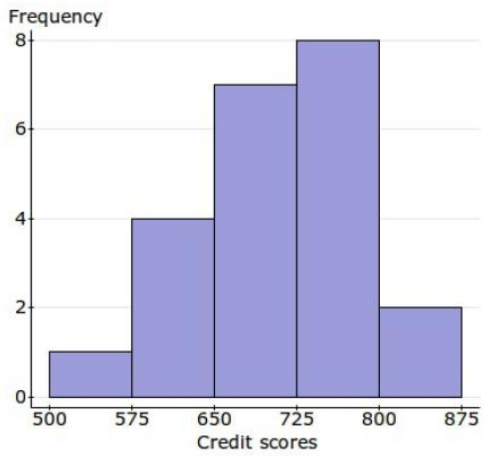
5. The histogram summarizes the magnitudes (Richter scale) of a sample of earthquakes. Refer to the histogram and comment on the relation between mean and median.



6. The amount spent on textbooks for one semester was recorded for a group of 200 students at your college. The mean expenditure was calculated to be 350 dollars. Suppose the textbook expenditure distribution is skewed to the right, which of the following values is most likely the value, in dollars, of the median of the textbook costs?

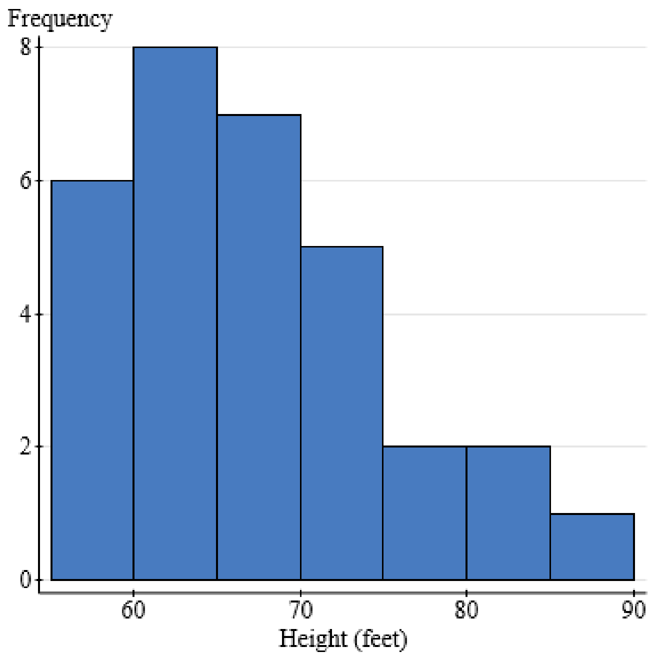
a. 450 b. 400 c. 350 d. 275

7. The histogram illustrates the credit scores of a group of people. Approximate the value of the mean.



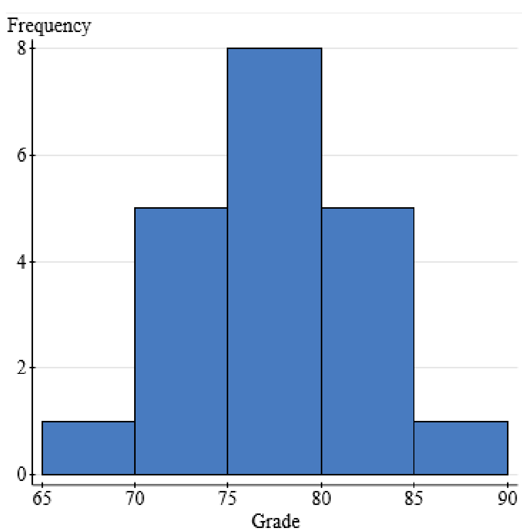
a. 715 b. 825 c. 600 d. 450

8. The histogram illustrates the heights of 31 black cherry trees. Approximate the value of the mean.



a. 60 b. 67 c. 75 d. 4.4

9. The histogram illustrates the test grades of 20 students. Which statement below is true?



- The mean is larger than the median.
- The mean is smaller than the median.
- The mean is the same as the median.

6.1 Answers

1. Mean = 31.7 pop ups, median = 33 pop ups
2.
   a. 16 students
   b. Mean = 8 students
   c. Median = 8 students
3.
   a. Mean = 29.79 dollars
   b. Median = 28.50 dollars

4.
   a. Mean = 3.02 min
   b. Median = 3.2 min

5. The mean will be smaller than the median because the data is skewed to the left.
6. d. 275
7. a. 715
8. b. 67
9. The mean is the same as the median.

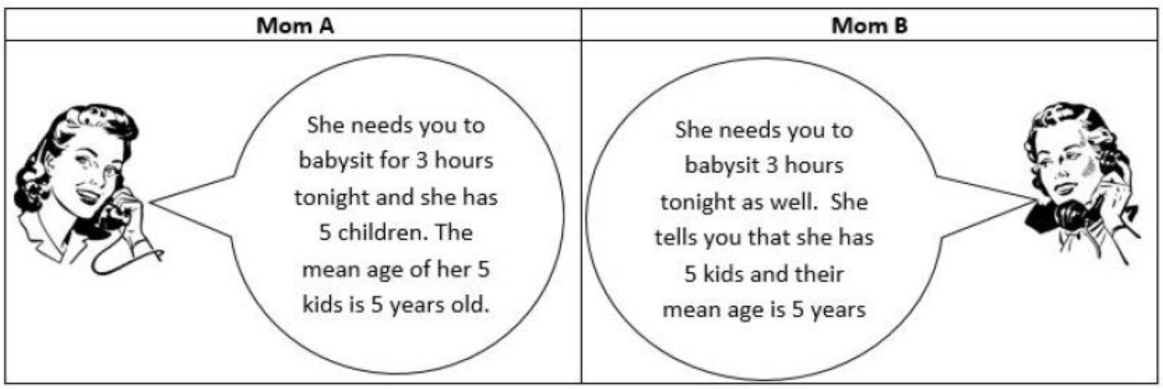
Section 6.2: Measures of Variation

Measures of variation (or spread) refers to a set of numerical summaries that describe the degree to which the data are spread out.

Why do we need them? Why is using measures of center not sufficient in describing data sets? To answer these kind of questions consider the following example.

Example 1. Babysitting

Suppose you are starting a new babysitting business. You have advertised in your neighborhood and it does not take long for calls to come in. You get calls from two moms, “Mom A” and “Mom B”. The information given by both moms is illustrated in the picture below.



As you can see the information provided is not enough to distinguish between the two families. It is highly unlikely that these two moms have kids that are the exact same ages. You politely ask the moms for the actual ages of their children.

Mom A	Mom B
1, 2, 4, 8, 10	3, 4, 5, 6, 7

We will use these two groups of data to find a new set of measures called *measures of variation: range and standard deviation*. These measures will help us quantify the age variation for each family and better understand our data.

Objective: Compute a range

Definition. *The* ***range*** *of a set of data is the difference between the largest value and the smallest value in the data set.*

$$\text{Range} = \text{Largest value} - \text{Smallest value}$$

The larger the range value, the more scattered the data.

Example 2. Babysitting

Find the range for our data:

Mom A	Mom B
1, 2, 4, 8, 10	3, 4, 5, 6, 7

Range of children's ages of Mom A $= 10 - 1 = 9$

Range of children's ages of Mom B $= 7 - 3 = 4$

Since the range value 9 is the higher of the two range values, we conclude that children's ages of Mom A have greater age variation than the children's ages of Mom B.

A disadvantage of using the range is that it only considers the maximum and the minimum values and ignores the rest of data.

Objective: Compute a standard deviation

We would prefer a measure of variation to quantify spread with respect to the center as well as to use all the data values in the set. This measure is the standard deviation.

Definition 3. *The **standard deviation** is the average distance the data values are from the mean.*

$$\sigma = \sqrt{\frac{\sum(x - \mu)^2}{N}}$$

Example 3. Babysitting

Consider the "Babysitting example" again. Find the standard deviation for Mom A children's ages.

Step 1: Start by setting up a table like the one below.

Step 2: Write each data value in the first column of the table. In column 2 find the difference between each age and the mean. Next square the values from column 2 and enter the results in column 3 (see the computations in the table below).

Mom A

Children's ages mean is $\mu = 5$

Age x	$Age - Mean$ $x - \mu$	$(Age - Mean)^2$ $(x - \mu)^2$
1	$1-5=-4$	$(-4)^2=16$
2	$2-5=-3$	$(-3)^2=9$
4	$4-5=-1$	$(-1)^2=1$
8	$8-5=3$	$(3)^2=9$
10	$10-5=5$	$(5)^2=25$

Step 3: Substitute the values in the formula for standard deviation and perform the computations.

$$\sigma = \sqrt{\frac{\sum(x-\mu)^2}{N}}$$ Standard deviation formula.

$$\sigma = \sqrt{\frac{16+9+1+9+25}{5}}$$ Add the values from column 3 of the table in the numerator. In the denominator put the number of children of Mom A.

$$\sigma = \sqrt{\frac{60}{5}}$$ Divide 60 by 5

$$\sigma = \sqrt{12}$$ Take square root of 12

$$\sigma \approx \mathbf{3.46}$$ This is the standard deviation of children's ages for Mom A.

Example 4. Babysitting

Find the standard deviation for Mom B children's ages.

We repeat the same steps as in previous example to find the standard deviation of children ages for Mom B.

Step 1: Set up the table.

Step 2: We write each data value in the first column of the table. In column 2 we find the difference between each age and the mean. Square the values from column 2 and enter the results in column 3 (see the computations in the table below).

Mom B

Children's ages mean is $\mu = 5$

Age x	$Age - Mean$ $x - \mu$	$(Age - Mean)^2$ $(x - \mu)^2$
3	$3-5=-2$	$(-2)^2=4$
4	$4-5=-1$	$(-1)^2=1$
5	$5-5=0$	$(0)^2=0$
6	$6-5=1$	$(1)^2=1$
7	$7-5=2$	$(2)^2=4$

Step 3: Substitute the values in the formula for standard deviation and perform the computations.

$$\sigma = \sqrt{\frac{\sum(x-\mu)^2}{N}}$$ Standard deviation formula.

$$\sigma = \sqrt{\frac{4+1+0+1+4}{5}}$$ Add the values from column 3 of the table in the numerator. In the denominator put the number of children of Mom B.

$$\sigma = \sqrt{\frac{10}{5}}$$ Divide 10 by 5.

$$\sigma = \sqrt{2}$$ Take square root of 2.

$$\sigma \approx \mathbf{1.41}$$ This is the standard deviation of children's ages for Mom B.

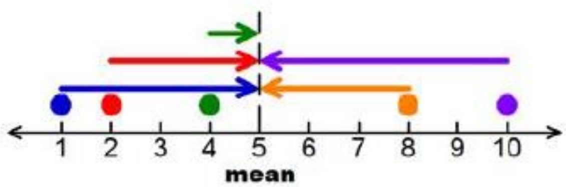
Objective: Compare distributions with different standard deviations

As mentioned previously, the standard deviation can be used to determine the variation (spread) of data. That is, the larger the standard deviation, the more the data values are spread. This also permits us to compare data sets with different standard deviations. Let us look at a few examples.

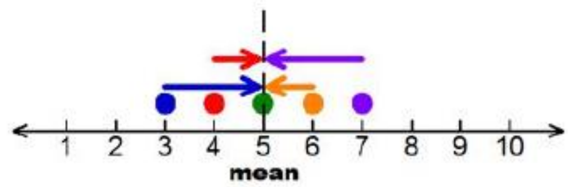
Example 5. Babysitting

We would like to compare the age variation (spread) between the two families. To visualize the idea of spread consider the dot plots below corresponding to our data sets.

Mom A



Mom B



Clearly, the data in the dot plot A is spread out more from the mean, while the data in the dot plot B is closer to the mean. Thus, the graphs show that Mom A children's ages are more spread out from the mean than the Mom B children's ages.

We can also compare the standard deviations calculated previously for each family. We have a standard deviation value of 3.46 which is larger than the standard deviation value of 1.41. Thus, we reach the same conclusion that Mom A children's ages have a greater age variation than Mom B children's ages.

Example 6. Emergency Room Waiting Time

Consider the following scenario:

The mean of waiting times in an emergency room is 90 minutes with a standard deviation of 12 minutes for people who are admitted for additional treatment. The mean waiting time for patients who are discharged after receiving treatment is 130 minutes with a standard deviation of 19 minutes.

In this example, by comparing the standard deviation values (13 min. and 19 min.), we can conclude that the waiting time for patients who are discharged after receiving treatment are more spread out than the waiting time for people who are admitted for additional treatment.

Example 7. Route to Work

You have two different routes to work each morning. The summary statistics are below.

Route 1: Mean = 14 minutes, Standard deviation = 4.3 minutes

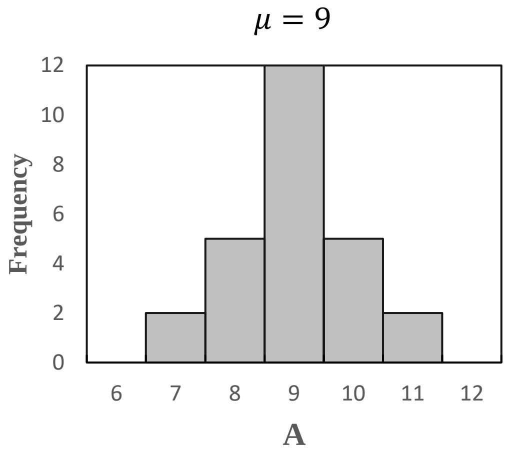
Route 2: Mean = 16 minutes, Standard deviation = 1.1 minutes

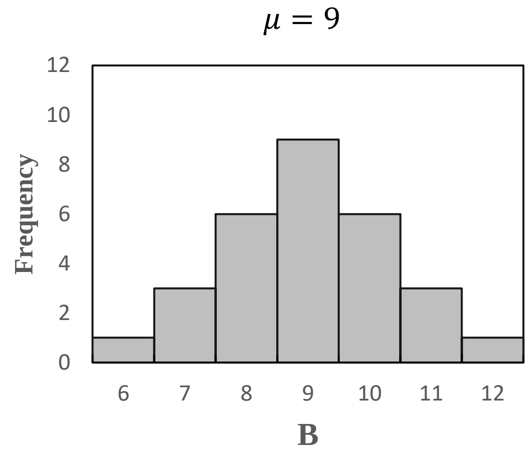
Which route offers you are more consistent commute time?

Route 2 has a standard deviation that is much smaller then Route 1. This will tells us that the route times they used to compute the summary statistics are more similar to one another. This ensures a more consistent commute.

Example 8. Bank Waiting Time

We can also determine which data set has a greater variation by looking at histograms. Consider the two histograms below which illustrate the waiting time (in minutes) of customers at two banks.





The data in histogram B is spread out more from the mean, while the data in histogram A is closer to the mean. This means that histogram B has a larger standard deviation than histogram A. Therefore, Bank A has more consistency in time.

6.2 Practice

1. The number of hours a student went to work after school last week are:

   1, 4, 3, 2, 6

   a. Find the range.
   b. Calculate the standard deviation.

2. The daily vehicle pass charge in dollars for six National parks are:

   15, 10, 6, 15, 20, 12

   a. Find the range.
   b. Calculate the standard deviation.

3. The range of scores on a statistics test was 40. The highest score was 99. What was the lowest score?

4. The weather station announced that the temperature fluctuates between a low of 80 F and a high of 93 F. Which measure of spread could be calculated using just this information? What is its value?

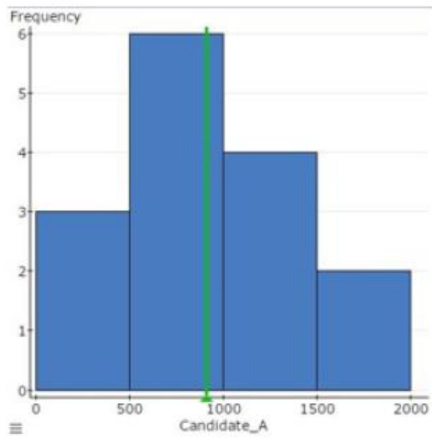
5. The table below shows the preparation tax return times (in hours) of two certified public accountants.

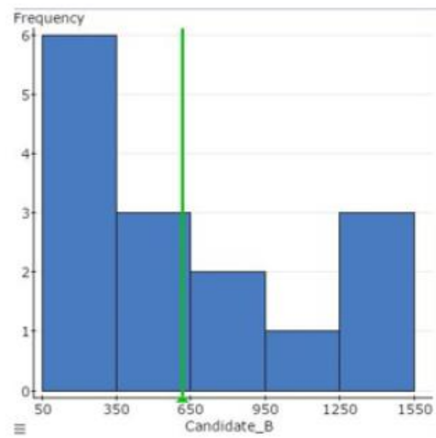
Accountant X	7	9	5	11	8
Accountant Z	5	11	14	3	7

   a. Find the range preparation time for each accountant.
   b. Calculate the standard deviations for each accountant.
   c. Who has the more consistent tax preparation time?

6. The average number of days construction workers miss per year is 11 with a standard deviation of 2.3. The average number of days factory workers miss per year is 8 with a standard deviation of 1.8. Which class of workers is more variable in terms of days missed?

7. The histograms below illustrate the contribution (in dollars) made to two presidential candidates in a recent election. Which histogram has a larger standard deviation?





8. The following table summarizes the wait times at a call center. Which center had a greater variation of wait times?

Call Center A	Call Center B
$n = 200$ calls	$n = 200$ calls
$\mu = 4.5$ minutes	$\mu = 3.2$ minutes
$\sigma = 1.2$ minutes	$\sigma = 1.6$ minutes

9. A movie director is trying to choose between 2 production companies. Company H spends an average of $88 million with a standard deviation of $15 million, while Company T spends an average of $95 million with a standard deviation of $9 million. Which company is more consistent in spending?

6.2 Answers

1.
   a. Range = 5
   b. $\sigma = 1.72$
2.
   a. Range = 14
   b. $\sigma = 4.40$

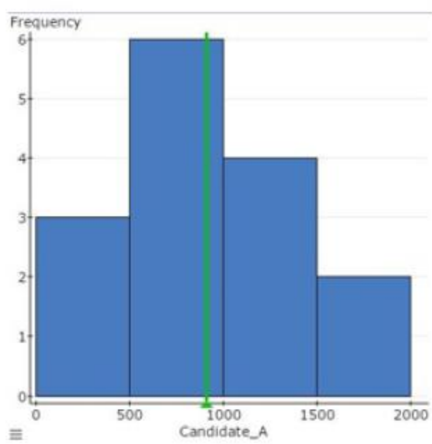
3. Lowest score is 59.

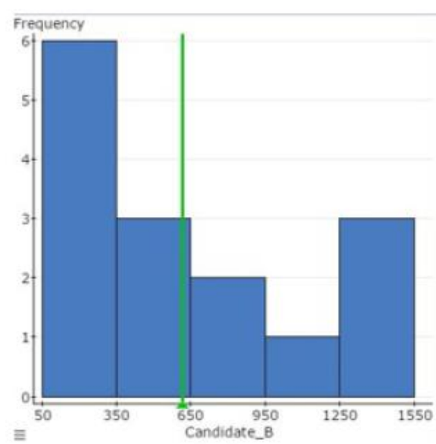
4. Range. The range is 13°F.

5.
   a. Range (Accountant X) = 6 hours Range (Accountant Z) = 11 hours
   b. σ (Accountant X) $= 2$ σ (Accountant Z) $= 4$
   c. Accountant X because their standard deviation is smaller.

6. The construction workers are more variable with their days missed because their standard deviation is larger. This means that their data is more varied.

7. The histograms below illustrate the contribution (in dollars) made to two presidential candidates in a recent election. Which histogram has a larger standard deviation?





Candidate B will have a larger standard deviation. You can tell because of the higher bars in the histogram are farther away from the mean (the green line). This means that more data are in the \$50 to \$349 and the \$1250 to \$1549 classifications.

8. Call center B does because it has a larger standard deviation.

9. Company T because it has a smaller standard deviation.

Section 6.3: Measures of Position

Measures of position are numbers showing the location of data values relative to the other values within a data set. They can be used to compare values from different data sets or to compare values within the same data set.

Why do we need them? Consider the following example.

Example 1. Graduate Record Exam

My friend Suzanne was applying to graduate schools and was required to take the Graduate Record Exam (GRE). The GRE is comprised of two sections: verbal and analytical. Suzanne knew that her verbal abilities were strong, but was worried about the analytical (math) section. Consequently, I helped her prepare for several weeks on the analytical section. On the test day, Suzanne received her raw scores immediately after completing the exam. Confused and worried, she called to tell me her scores:

Verbal section	Analytical section
380	420

To be accepted in the programs she was interested in, Suzanne needed a high score on the verbal section. I assured her that she shouldn't panic because these were only her raw scores. She needs to wait until the end of the testing period to receive the summary of students' scores in order to have a better idea of how she really did on the exam.

After waiting several long weeks a letter from the testing company came with the following summary:

Section	**Mean**	**Standard Deviation**
Verbal	356	12
Analytical	400	20

Based on the mean, it appears as though the verbal section (mean 356) was a little more difficult than the analytical section (mean 400). Perhaps this explains her "reverse" performance. Also notice Suzanne's two scores are measured on different scales (standard deviations).

Objective: Determine and interpret z-scores.

The best way to evaluate how my friend did on these sections is to compare her scores to all of her peers by finding out how many standard deviations her scores are from the mean scores.

In other words, we need to standardize the values by converting them to $z-$scores.

Definition 2. *The $z-$score (or standardized value) indicates the number of standard deviations a given value x is above or below the mean.*

$$z = \frac{x-\mu}{\sigma}$$

A positive $z-$score means the data value is situated above the mean.

A zero $z-$score means the data value is situated exactly at the mean.

A negative $z-$score means the data value is situated below the mean.

Example 2. Find Suzanne's $z-$score on the verbal section. Recall, her raw score was 380, the mean was 356, and the standard deviation was 12.

$z_{\text{verbal}} = \dfrac{x-\mu}{\sigma}$	$z-$score formula Insert the score (x) and the mean (μ) for the verbal section
$z_{\text{verbal}} = \dfrac{380-356}{12}$	In the denominator insert the standard deviation (σ)
$z_{\text{verbal}} = \dfrac{24}{12}$	Divide 24 by 12
$z_{\text{verbal}} = 2$	This is her $z-$score for the verbal section.

Interpretation: Her verbal score was 2 standard deviations above the mean.

Example 3. Find Suzanne's $z-$score on the analytical section. Recall, her raw score was 420, the mean was 400, and the standard deviation was 20.

$z_{\text{analytical}} = \dfrac{x-\mu}{\sigma}$	$z-$score formula Insert the score (x) and the mean (μ) for the analytical section
$z_{\text{analytical}} = \dfrac{420-400}{20}$	In the denominator insert the standard deviation (σ)
$z_{\text{analytical}} = \dfrac{20}{20}$	Divide 20 by 20
$z_{\text{analytical}} = 1$	This is her $z-$score for the analytical section.

Interpretation: Her analytical score was 1 standard deviation above the mean.

Conclusion: Suzanne is certainly above average on both sections. We can quickly see that with the positive z-scores.

Example 4. On which section do you think she performed better?

The $z-$scores show that Suzanne's score on the verbal section is 2 standard deviations above the mean while her score on the analytical section is 1 standard deviation above the mean. Therefore, Suzanne performed better on the verbal section because it has a larger z-score.

Example 5. Computing a z-score

Liv's web page viewing varies from day to day with mean 200 pages and standard deviation 85 pages. Yesterday Liv viewed 305 web pages. What is her z-score?

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{305\,pages - 200\,pages}{85\,pages}$$

$$z = \frac{105\,\cancel{pages}}{85\,\cancel{pages}}$$

$$z = 1.24$$

Example 6. Using z-scores to compare groups

Angie (10 yr old) and Beth (17 yr old) are sisters and wanted to know who had the fastest time for the 50 meter free-style when compared to their own team.

	Swimmer Times	Team mean	Team standard deviation
Angie	34.6 sec	37.3 sec	2.0 sec
Beth	27.3 sec	30.1 sec	1.4 sec

$$z_{\text{Angie}} = \frac{x - \mu}{\sigma} \qquad z_{\text{Beth}} = \frac{x - \mu}{\sigma}$$

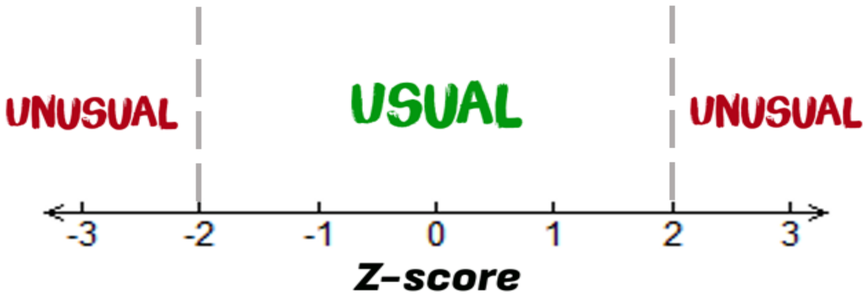
$$z_{\text{Angie}} = \frac{34.6sec - 37.3sec}{2.0sec} \qquad z_{\text{Beth}} = \frac{27.3sec - 30.1sec}{1.4sec}$$

$$z_{\text{Angie}} = \frac{-2.7\cancel{sec}}{2.0\cancel{sec}} \qquad z_{\text{Beth}} = \frac{-2.8\cancel{sec}}{1.4\cancel{sec}}$$

$$z_{\text{Angie}} = -1.35 \qquad z_{\text{Beth}} = -2.8$$

Both girls had great swims that are less than their team's average. Once we standardize their times using their whole team's results, we can see that Beth had a faster swim time. Her z-score of -2.8 is farther from 0 than Angie's z-score of -1.35.

Objective: Unusual z-score



Remember, z-scores measure the number of standard deviations your data value is from the mean. A rule of thumb that we use in statistics is demonstrated in the diagram above.

- If a z-score is greater than 2 it is considered unusual. (Unusually high)
- If a z-score is less than -2 it is considered unusual. (Unusually low)
- If a z-score is between -2 and 2 it is considered usual.

Example 7. Determining if a data value is unusual using a z-score

A light bulb company claims that the average lifetime of its bulbs is 1000 hours with a standard deviation of 75 hours. If your light bulb only lasts 775 hours, is that an unusually short time to last, relatively speaking?

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{775\ hours - 1000\ hours}{75\ hours}$$

$$z = \frac{-225\ \cancel{hours}}{75\ \cancel{hours}}$$

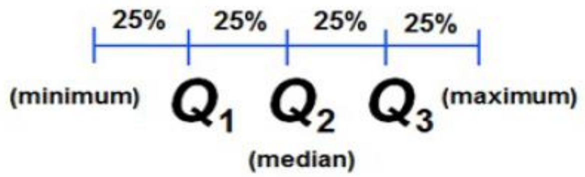
$$z = -3$$

Yes, because the z-score is -3.

Objective: Determine and interpret quartiles.

Quartiles are also measures of position. For example, if your child's pediatrician tells you that your child's height is at the third quartile (Q_3) for his/her age this means 75 percent of children of the same age as your child are of the same height or shorter than your child.

Definition 6. *As the name implies,* ***quartiles*** *of a data set divide the data set into four equal parts, each containing 25% of the data.*



- *The first quartile (Q_1) separates the bottom 25% of sorted values from the top 75%.*
- *The second quartile (Q_2) which is also the median separates the bottom 50% of sorted values from the top 50%.*
- *The third quartile (Q_3) separates the bottom 75% of sorted values from the top 25%.*

Steps to find quartiles:

1. Arrange the data in order of lowest to highest.
2. Find the median which is also the second quartile (Q_2)
3. Find the middle of the first half of the data which is the first quartile (Q_1)
4. Find the middle of the second half of the data which is the third quartile (Q_3)

Example 8. Commute distance to school

The following data is the commute distance (in miles) of a group of students. Find the quartiles for this data.

1	2	4	5	6	4	1	12	10	30	1	6
10	6	5	9	7	5	8	17	25	35	12	10

First, we order the values:

1 1 1 2 4 4 5 5 5 6 6 6 7 8 9 10 10 10 12 12 17 25 30 35

Second, we find the median (Q_2):

Position of the median $= \frac{24+1}{2} = 12.5$

We will need to determine what number is exactly between the 12th and 13th data value. The midpoint formula will help us determine this.

$$Q_2 = \text{Median} = \frac{6+7}{2} = 6.5$$

1 1 1 2 4 4 5 5 5 6 6 [6 7] 8 9 10 10 10 12 12 17 25 30 35

$Q_2 = 6.5$

Next, we find the middle of the first half of the data which is the first quartile (Q_1) and the middle of the second half of the data which is Q_3 (see picture below). Each of the halves have 12 data values so the middle is between each of their 6th and 7th data values.

first half of the data | *second half of the data*

1 1 1 2 4 (4) (5) 5 5 6 6 6 | 7 8 9 10 10 (10) (12) 12 17 25 30 35

$Q_1 = 4.5$ $Q_3 = 11$

Thus, the quartiles for the data in this example are 4.5, 6.5, and 11, respectively.

Example 9. Ages of students in an Algebra class.

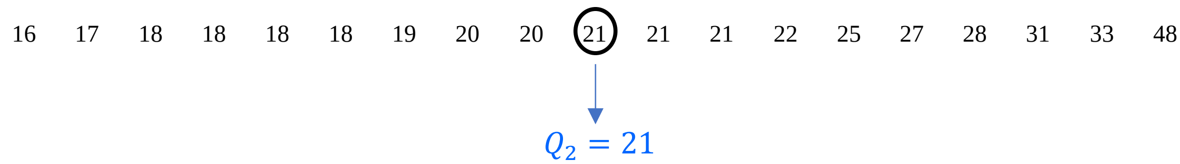
The following data represents the ages of 19 students taking a winter session of *Intermediate Algebra*. **Note**: These ages are already sorted.

16 17 18 18 18 18 19 20 20 21 21 21 22 25 27 28 31 33 48

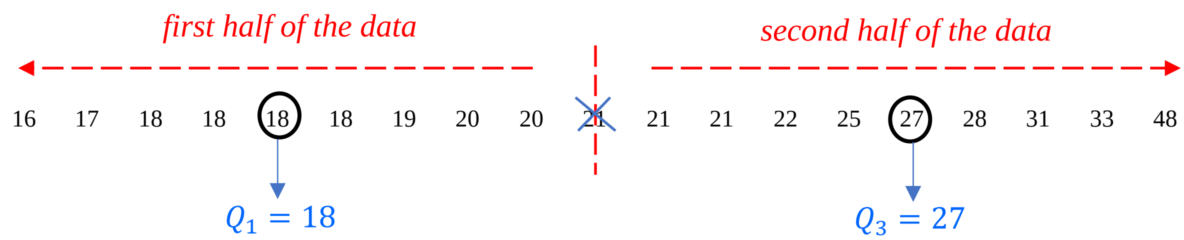
Find the median (Q_2):

Position of the median $\frac{19+1}{2} = 10^{th}$

10th value



Find Q_1(the middle of the first half of the data) and Q_3 (the middle of the second half of the data.



Thus, the quartiles for the data in this example are 18, 21, and 27, respectively.

Objective: Compute the five – number summary

Definition 11. ***A five number summary** consists of:*

i. *The minimum (smallest observation)*
ii. *The first quartile* $(Q1)$
iii. *The median* $(Q2)$
iv. *The third quartile* $(Q3)$
v. *The maximum (largest observation)*

Example 10. Write the 5 – number summary for the commute distance to school example.

$$Minimum = 1 \quad Q_1 = 4.5 \quad Q_2 = 6.5 \quad Q_3 = 11 \quad Maximum = 35$$

Example 11. Write the 5 – number summary for the ages of the algebra students example

$$Minimum = 16 \quad Q_1 = 18 \quad Q_2 = 21 \quad Q_3 = 27 \quad Maximum = 48$$

6.3 Practice

1. Suppose the average score on the commercial driver's license test in Maryland is 79 with a standard deviation of 5. Find the corresponding $z-$scores for each raw score.

   a. 89 b. 79 c. 77 d. 92 e. 81

2. The average cost in dollars of a wedding in United States (not including the cost for a honeymoon) is $26,650 with a standard deviation of $2,650. Find the corresponding $z-$scores for each raw score.

   a. $29,300 b. $10,750 c. $32,745

3. The z-score for the life span of your Apple iPhone is 0.78. Is your phone performing better than average or below average?

4. Among males, Martin's weight has a *z*-score of -1.90. Is Martin heavier or lighter than most men?

5. Suppose that the number of complaints in a month against police department B has a mean of 14 with a standard deviation of 2. This past month they had 11 complaints. Is this unusual relative to their other months?

6. After two years of school, a student who attends a university has a loan of $19,000 where the average debt is 16,000 and the standard deviation 1,500. After two years of school, another student who attends a college has a loan of 6,250 where the average debt is 6,000 and the standard deviation 500. Which student has a higher debt in relationship to his or her peers?

7. The tallest living man has a height of 243 cm. The tallest living woman is 234 cm tall. Heights of men have a mean of 173 cm and a standard deviation of 7 cm. Heights of women have a mean of 162 cm and a standard deviation of 5 cm. Relative to the population of the same gender, find who is taller.

8. Data from the last ten years shows that the average high school male has a personal best long jump of 18.9 feet with a standard deviation of 1.9 feet. The average high school female has a personal best jump of 16.5 feet with a standard deviation of 2.2 feet. Jason's longest jump is 20.2 feet. Sara's longest jump is 19 feet. Which athlete has a more extraordinary jump?

9. Juneau Alaska has an average snowfall in January of 24 inches with a standard deviation of 3.5 inches. Milwaukee Wisconsin has an average snowfall in January of 15 inches with a standard deviation of 2.8 inches. In January 2020, Juneau had 29.2 inches of snow and Milwaukee had 16.2 inches of snow. Which city is having a more extraordinary snowfall in January?

10. ***Multiple Choice***. Leon has a z-score of -1.3 for his Test 1 grade. Which of the following correctly interprets his z-score?
    a. Leon scored 1.3 points lower than the mean.
    b. Leon scored 1.3 points above the mean.
    c. Leon scored 1.3 standard deviations below the mean.
    d. Leon performed 1.3% lower than the class.

11. A group of 14 diners was asked how much they would pay (in dollars) for a meal. Their responses were (in dollars):

5	7	6	8	11	9	25	12	9	16	15	10	13	8

Identify the five number summary for this data.

12. The data below are the number of wins in the regular season games for the Baltimore Ravens from 2004 to 2019.

10	6	13	5	11	9	12	12	10	8	10	5	8	9	10	14

Identify the five number summary for this data.

13. The data below represents the number of wins in the 2019 regular season for 15 Major League Baseball teams. Identify the five number summary for this data.

Team	**Number of wins**
New York Yankees	103
Tampa Bay Rays	96
Boston Red Sox	84
Toronto Blue Jays	67
Baltimore Orioles	54
Minnesota Twins	101
Cleveland Indians	93
Chicago White Sox	72
Kansas City Royals	59
Detroit Tigers	47
Houston Astros	107
Oakland Athletics	97
Texas Rangers	78
Los Angeles Angels	72
Seattle Mariners	68

14. The following data is the pulse rates of 17 thirty year old females. Identify the five number summary for this data.

82	85	60	69	57	79	91	63	77	70	64	86	88	72	58	76	69

6.3 Answers

1. a. $z = 2$
   b. $z = 0$
   c. $z = -0.4$
   d. $z = 2.6$
   e. $z = 0.4$
2. a. $z = 1$
   b. $z = -6$
   c. $z = 2.30$
3. Above average because the z-score is positive.
4. Lighter than most men because his z-score is negative.

5. $z = \frac{11-14}{2} = -1.5$
   No, it is not unusually low because the z-score is not less than -2.

6. $z = 2$ for the student who attends the university
   $z = 0.5$ for the student who attends the college
   The student attending the university has a higher debt in relationship to his or her peers. This is because his or her z-score is larger.
7. $z_{\text{man}} = 10$
   $z_{\text{woman}} = 14.4$
   The woman is taller relative to the population of the same gender. This is because her $z-$score is larger.

8.

$$z_{Jason} = \frac{20.2 - 18.9}{1.9} = 0.68$$

$$z_{Sara} = \frac{19 - 16.5}{2.2} = 1.14$$

Sarah has a more extraordinary jump because her z-score is larger.

9.

$$z_{Juneau} = \frac{29.2 - 24}{3.5} = 1.7$$

$$z_{Milwaukee} = \frac{16.2 - 15}{2.8} = 0.43$$

Juneau had a more extraordinary snowfall in January 2020 because its z-score is larger.

10. c. Leon scored 1.3 standard deviations below the mean.

11. $Minimum = \$5 \quad Q_1 = \$8 \quad Median = \$9.50 \quad Q_3 = \$13 \quad Maximum = \$25$

12. $Minimum = 5 \quad Q_1 = 8 \quad Median = 10 \quad Q_3 = 11.5 \quad Maximum = 14$

13. $Minimum = 47\ wins,\ Q_1 = 67\,wins,\ Median = 78\,wins,\ Q_3 = 97\,wins,$ $Maximum = 107\,wins$

14. $Minimum = 57 \quad Q_1 = 63.5 \quad Median = 72 \quad Q_3 = 83.5 \quad Maximum = 91$