Section 4.3: Factoring Trinomials When the Leading Coefficient is One

Objective: Factor trinomials when the leading coefficient or the coefficient of x^2 is 1.

Factoring polynomials with three terms, or factoring trinomials, is the most important type of factoring to be mastered. Since factoring can be thought of as the reverse of multiplication, we will start with a multiplication problem and look at how we can reverse the process.

Example 1. Multiply and simplify.

$(x+6)(x-4)$	Distribute $(x+6)$ through second parentheses
$x(x+6)-4(x+6)$	Distribute each monomial through parentheses
$x^2+6x-4x-24$	Combine like terms
$x^2+2x-24$	Our Solution

You may notice that if you reverse the last three steps, the process looks like grouping. This is because it is grouping! The GCF of the left two terms is x and the negative of the GCF of the second two terms is -4. The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This is shown in the following example, which is the same problem worked backwards:

Example 2. Factor completely.

$x^2+2x-24$	Split middle term into $+6x-4x$
$x^2+6x-4x-24$	Grouping; GCF on left is x; negative of GCF on right is -4
$x(x+6)-4(x+6)$	$(x+6)$ is the same; factor out this GCF
$(x+6)(x-4)$	Our Solution

The trick to making these problems work is in how we split the middle term. Why did we pick $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that works! So, how do we know what is the one combination that works? To find the correct way to split the middle term, we find a pair of numbers that multiply to obtain the last term in the trinomial and also sum to the number that is the coefficient of the middle term of the trinomial. In the previous example that would mean we wanted to multiply to -24 and sum to 2. The only numbers that can do this are 6 and -4 ($6\cdot -4=-24$ and $6+(-4)=2$). This process is shown in the next few examples.

Example 3. Factor completely.

$x^2+9x+18$	Multiply to 18; sum to 9
$x^2+6x+3x+18$	6 and 3; split the middle term
$x(x+6)+3(x+6)$	Factor by grouping

$(x+6)(x+3)$ Our Solution

Example 4. Factor completely.

x^2-4x+3	Multiply to 3; sum to -4
$x^2-3x-x+3$	-3 and -1; split the middle term
$x(x-3)-1(x-3)$	Factor by grouping
$(x-3)(x-1)$	Our Solution

Example 5. Factor completely.

$x^2-8x-20$	Multiply to -20; sum to -8
$x^2-10x+2x-20$	-10 and 2; split the middle term
$x(x-10)+2(x-10)$	Factor by grouping
$(x-10)(x+2)$	Our Solution

Often when factoring, we have two variables. These problems solve just like problems with one variable, using the coefficients to decide how to split the middle term.

Example 6. Factor completely.

$a^2-9ab+14b^2$	Multiply to 14; sum to -9
$a^2-7ab-2ab+14b^2$	-7 and -2; split the middle term
$a(a-7b)-2b(a-7b)$	Factor by grouping
$(a-7b)(a-2b)$	Our Solution

Warning! Notice that it is very important to be aware of negatives, as we find the pair of numbers we will use to split the middle term. Consider the following example, done incorrectly, ignoring negative signs:

Factor x^2+5x-6	Multiply to 6; sum to 5
$x^2+2x+3x-6$	2 and 3; split the middle term
$x(x+2)+3(x-2)$	Factor by grouping
???	Binomials do not match!

Because we did not use the negative sign with the 6 to find our pair of numbers, the binomials did not match and grouping was not able to work at the end. Now the problem will be done correctly:

Example 7. Factor completely.

x^2+5x-6	Multiply to -6; sum to 5
$x^2+6x-1x-6$	6 and -1; split the middle term
$x(x+6)-1(x+6)$	Factor by grouping
$(x+6)(x-1)$	Our Solution

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1; our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it, however, when we have no number (technically we have a 1) in front of x^2. In all of the problems we have factored in this lesson, there is no number written in front of x^2. If this is the case, then we can use this shortcut. This is shown in the next few examples.

Example 8. Factor completely.

$x^2-7x-18$	Multiply to -18; sum to -7
	-9 and 2; write the factors
$(x-9)(x+2)$	Our Solution

Example 9. Factor completely.

$m^2-mn-30n^2$	Multiply to -30; sum to -1
	5 and -6; write the factors; don't forget the second variable
$(m+5n)(m-6n)$	Our Solution

It is possible to have a problem that does not factor. If there is no combination of numbers that multiplies and adds up to the correct numbers, then we say we cannot factor the polynomial or we say the polynomial is prime. This is shown in the following example.

Example 10. Factor completely.

x^2+2x+6	Multiply to 6; sum to 2
$1\cdot 6$ and $2\cdot 3$	Only possibilities to multiply to 6; none sum to 2
Prime	Our Solution

When factoring, it is important not to forget about the GCF. If all of the terms in a problem have a common factor, we will want to first factor out the GCF before we attempt using any other method. The next three examples illustrate this technique:

Example 11. Factor completely.

$3x^2-24x+45$	GCF of all terms is 3; factor out 3
$3\left(x^2-8x+15\right)$	Multiply to 15; sum to -8
	-5 and -3; write the factors
$3\left(x^2-8x+15\right)$	
$3(x-5)(x-3)$	Our Solution

Example 12. Factor completely.

$4x^2y-8xy-32y$	GCF of all terms is $4y$; factor out $4y$
$4y(x^2-2x-8)$	Multiply to -8; sum to -2
	-4 and 2; write the factors
$4y\,(x-4)\,(x+2)$	Our Solution

Example 13. Factor completely.

$7a^4b^2+28a^3b^2-35a^2b^2$	GCF of all terms is $7a^2b^2$; factor out $7a^2b^2$
$7a^2b^2(a^2+4a-5)$	Multiply to -5; sum to 4
	-1 and 5; write the factors
$7a^2b^2(a-1)\,(a+5)$	Our Solution

Again it is important to comment on the shortcut of jumping right to the factors. This only works if there is no written coefficient of x^2; that is, the leading coefficient is understood to be 1. Be careful not to use this shortcut on all factoring problems!

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables we are solving for, just as codes used letters to represent an unknown message.

4.3 Practice

Factor each expression completely.

1) $p^2+17p+72$
2) x^2+x-72
3) n^2-9n+8
4) x^2+x-30
5) $x^2-9x-10$
6) $x^2+13x+40$
7) $b^2+12b+32$
8) $b^2-17b+70$
9) $x^2+3x-70$
10) $x^2+3x-18$
11) $n^2-8n+15$
12) $a^2-6a-27$
13) $p^2+15p+54$
14) $p^2+7p-30$
15) c^2-4c+9
16) $m^2-15mn+50n^2$
17) $u^2-8uv+15v^2$
18) $m^2-3mn-40n^2$
19) $m^2+2mn-8n^2$
20) $x^2+10xy+16y^2$
21) $x^2-11xy+18y^2$
22) $u^2-9uv+14v^2$
23) $x^2+xy-12y^2$
24) $x^2+14xy+45y^2$
25) $x^2+4xy-12y^2$
26) $4x^2+52x+168$
27) $5a^2+60a+100$
28) $7w^2+5w-35$
29) $6a^2+24a-192$
30) $5v^2+20v-25$
31) $6x^2+18xy+12y^2$
32) $5m^2+30mn-80n^2$

33) $6x^2+96xy+378y^2$

34) $6m^2-36mn-162n^2$

35) $n^2-15n+56$

36) $5n^2-45n+40$

4.3 Answers

1) $(p+9)(p+8)$
2) $(x-8)(x+9)$
3) $(n-8)(n-1)$
4) $(x-5)(x+6)$
5) $(x+1)(x-10)$
6) $(x+5)(x+8)$
7) $(b+8)(b+4)$
8) $(b-10)(b-7)$
9) $(x-7)(x+10)$
10) $(x-3)(x+6)$
11) $(n-5)(n-3)$
12) $(a+3)(a-9)$
13) $(p+6)(p+9)$
14) $(p+10)(p-3)$
15) Prime
16) $(m-5n)(m-10n)$
17) $(u-5v)(u-3v)$
18) $(m+5n)(m-8n)$
19) $(m+4n)(m-2n)$
20) $(x+8y)(x+2y)$
21) $(x-9y)(x-2y)$
22) $(u-7v)(u-2v)$
23) $(x-3y)(x+4y)$
24) $(x+5y)(x+9y)$
25) $(x+6y)(x-2y)$
26) $4(x+7)(x+6)$
27) $5(a+10)(a+2)$
28) Prime
29) $6(a-4)(a+8)$
30) $5(v-1)(v+5)$
31) $6(x+2y)(x+y)$
32) $5(m-2n)(m+8n)$
33) $6(x+9y)(x+7y)$
34) $6(m-9n)(m+3n)$
35) $(n-8)(n-7)$
36) $5(n-8)(n-1)$