Section 4.4: Factoring Special Forms of Polynomials

Objective: Identify and factor special forms of polynomials including a difference of two squares and perfect square trinomials.

When factoring, there are a few special forms of polynomials that, if we can recognize them, help us factor polynomials. The first is one we have seen before. When multiplying special products, we found that a sum and a difference could multiply to a difference of two squares. Here, we will use this special product to help us factor the difference of two squares.

Difference of Two Squares: $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

If we are subtracting two squares, then the expression will always factor to the sum and difference of the square roots.

Example 1. Factor completely.

$x^2 - 16$	Subtracting two squares; the square roots are x and 4
$(x+4)(x-4)$	Our Solution

Example 2. Factor completely.

$9a^2 - 25b^2$	Subtracting two squares; the square roots are $3a$ and $5b$
$(3a+5b)(3a-5b)$	Our Solution

In the next example, we will see that, generally, the sum of two squares cannot be factored.

Example 3. Factor completely.

$x^2 + 36$	No bx term; we use $0x$.
$x^2 + 0x + 36$	Multiply to 36; sum to 0
$1\cdot 36, 2\cdot 18, 3\cdot 12, 4\cdot 9, 6\cdot 6$	No combinations that multiplies to 36 and sum to 0
Prime (cannot be factored)	Our Solution

It turns out that a sum of two squares is generally considered to be prime when the exponent is 2. If the exponent is greater than 2, then factoring the sum of two squares will go beyond the scope of this course.

Sum of Two Squares: $\boldsymbol{a^2 + b^2}$ **= prime (generally cannot be factored)**

A great example where we see a sum of two squares and a difference of two squares together would be factoring a difference of fourth powers. Because the square root of a fourth power is a square $\left(\sqrt{a^4} = a^2\right)$, we can factor a difference of fourth powers, just like we factor a difference of two squares, to a sum and difference of the square roots. This will give us two factors: one which will be a prime sum of two squares; and a second that will be a difference of two squares, which we can factor again. This is shown in the following two examples.

Example 4. Factor completely.

$a^4 - b^4$	Difference of two squares with square roots a^2 and b^2
$(a^2 + b^2)\,(a^2 - b^2)$	The first factor is prime; the second is a difference of two squares with square roots a and b
$(a^2 + b^2)\,(a + b)\,(a - b)$	Our Solution

Example 5. Factor completely.

$x^4 - 16$	Difference of two squares with square roots x^2 and 4
$(x^2 + 4)\,(x^2 - 4)$	The first factor is prime; the second is a difference of two squares with square roots x and 2
$(x^2 + 4)\,(x + 2)\,(x - 2)$	Our Solution

Another factoring formula is the perfect square trinomial. We had a shortcut for squaring binomials, which can be reversed to help us factor a perfect square trinomial.

Perfect Square Trinomial: $\boldsymbol{a^2 + 2ab + b^2 = (a + b)^2}$

Here is how to recognize a perfect square trinomial:

(1) The first term is a square of a monomial or an integer.
(2) The middle term is two times the product of the square root of the first and last terms.
(3) The third term is a square of a monomial or an integer.

Then, we can factor a perfect square trinomial using the square roots of the first and last terms and the sign from the middle term. This is shown in the following examples.

Example 6. Factor completely.

$x^2 - 6x + 9$	$x^2 = (x)^2;\ 6x = 2(x)(3);\ 9 = (3)^2$
$(x)^2 - 2 \cdot x \cdot 3 + (3)^2$	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$(x - 3)^2$	Our Solution

Example 7. Factor completely.

$4x^2 + 20xy + 25y^2$	$4x^2 = (2x)^2;\ 20xy = 2(2x)(5y);\ 25y^2 = (5y)^2$
$(2x)^2 + 2 \cdot 2x \cdot 5y + (5y)^2$	Perfect square trinomial; use square roots from first and last terms and sign from the middle
$(2x + 5y)^2$	Our Solution

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. For example, problems would be written as: "three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou." If g represents a sheaf of good crop, m represents a sheaf of mediocre crop, and b respresents a sheaf of bad crop, then we can say that the polynomial $3g+2m+b$ equals 29.

The following table sumarizes all of the formulas that we can use to factor special forms of polynomials.

Factoring Special Forms of Polynomials

Difference of Two Squares:	$a^2-b^2=(a+b)(a-b)$
Sum of Two Squares:	$a^2+b^2=$ Prime (generally cannot be factored)
Perfect Square Trinomial:	$a^2+2ab+b^2=(a+b)^2$

As always, when factoring special forms of polynomials, it is important to check for a GCF first. Only after checking for a GCF should we use the special products used in the factoring formulas. This is shown in the following examples.

Example 8. Factor completely.

$72x^2-2$	GCF is 2
$2(36x^2-1)$	Difference of two squares; square roots are $6x$ and 1
$2(6x+1)(6x-1)$	Our Solution

Example 9. Factor completely.

$48x^2y-24xy+3y$	GCF is $3y$
$3y(16x^2-8x+1)$	$16x^2=(4x)^2; 8x=2(4x)(1); 1=(1)^2;$ perfect square trinomial
	$16x^2-8x+1=(4x)^2-2\cdot 4x\cdot 1+(1)^2$
	Use square roots from first and last terms and sign from the middle
$3y(4x-1)^2$	Our Solution

4.4 Practice

Factor each expression completely.

1) $r^2 - 16$
2) $x^2 - 9$
3) $v^2 - 25$
4) $x^2 - 1$
5) $p^2 - 4$
6) $4v^2 - 1$
7) $9k^2 - 4$
8) $9a^2 - 1$
9) $3x^2 - 27$
10) $5n^2 - 20$
11) $16x^2 - 36$
12) $125x^2 + 45y^2$
13) $18a^2 - 50b^2$
14) $4m^2 + 64n^2$
15) $a^2 - 2a + 1$
16) $k^2 + 4k + 4$
17) $x^2 + 6x + 9$
18) $n^2 - 8n + 16$
19) $x^2 - 6x + 9$
20) $k^2 - 4k + 4$
21) $25p^2 - 10p + 1$
22) $x^2 + 2x + 1$
23) $25a^2 + 30ab + 9b^2$
24) $x^2 + 8xy + 16y^2$
25) $4a^2 - 20ab + 25b^2$
26) $49x^2 + 36y^2$
27) $8x^2 - 24xy + 18y^2$
28) $20x^2 + 20xy + 5y^2$
29) $a^4 - 81$
30) $x^4 - 256$
31) $16 - z^4$
32) $n^4 - 1$
33) $x^4 - y^4$

34) $16a^4 - b^4$
35) $m^4 - 81b^4$
36) $81c^4 - 16d^4$
37) $18m^2 - 24mn + 8n^2$
38) $w^4 + 225$

4.4 Answers

1) $(r+4)(r-4)$
2) $(x+3)(x-3)$
3) $(v+5)(v-5)$
4) $(x+1)(x-1)$
5) $(p+2)(p-2)$
6) $(2v+1)(2v-1)$
7) $(3k+2)(3k-2)$
8) $(3a+1)(3a-1)$
9) $3(x+3)(x-3)$
10) $5(n+2)(n-2)$
11) $4(2x+3)(2x-3)$
12) $5(25x^2+9y^2)$
13) $2(3a+5b)(3a-5b)$
14) $4(m^2+16n^2)$
15) $(a-1)^2$
16) $(k+2)^2$
17) $(x+3)^2$
18) $(n-4)^2$
19) $(x-3)^2$
20) $(k-2)^2$
21) $(5p-1)^2$
22) $(x+1)^2$
23) $(5a+3b)^2$
24) $(x+4y)^2$
25) $(2a-5b)^2$
26) Prime
27) $2(2x-3y)^2$
28) $5(2x+y)^2$
29) $(a^2+9)(a+3)(a-3)$
30) $(x^2+16)(x+4)(x-4)$
31) $(4+z^2)(2+z)(2-z)$
32) $(n^2+1)(n+1)(n-1)$
33) $(x^2+y^2)(x+y)(x-y)$

34) $(4a^2+b^2)(2a+b)(2a-b)$

35) $(m^2+9b^2)(m+3b)(m-3b)$

36) $(9c^2+4d^2)(3c+2d)(3c-2d)$

37) $2(3m-2n)^2$

38) Prime