Section 6.3: Measures of Position

Measures of position are numbers showing the location of data values relative to the other values within a data set. They can be used to compare values from different data sets or to compare values within the same data set.

Why do we need them? Consider the following example.

Example 1. Graduate Record Exam

My friend Suzanne was applying to graduate schools and was required to take the Graduate Record Exam (GRE). The GRE is comprised of two sections: verbal and analytical. Suzanne knew that her verbal abilities were strong, but was worried about the analytical (math) section. Consequently, I helped her prepare for several weeks on the analytical section. On the test day, Suzanne received her raw scores immediately after completing the exam. Confused and worried, she called to tell me her scores:

Verbal section	Analytical section
380	420

To be accepted in the programs she was interested in, Suzanne needed a high score on the verbal section. I assured her that she shouldn't panic because these were only her raw scores. She needs to wait until the end of the testing period to receive the summary of students' scores in order to have a better idea of how she really did on the exam.

After waiting several long weeks a letter from the testing company came with the following summary:

Section	**Mean**	**Standard Deviation**
Verbal	356	12
Analytical	400	20

Based on the mean, it appears as though the verbal section (mean 356) was a little more difficult than the analytical section (mean 400). Perhaps this explains her "reverse" performance. Also notice Suzanne's two scores are measured on different scales (standard deviations).

Objective: Determine and interpret z-scores.

The best way to evaluate how my friend did on these sections is to compare her scores to all of her peers by finding out how many standard deviations her scores are from the mean scores.

In other words, we need to standardize the values by converting them to $z-$scores.

Definition 2. *The $z-$score (or standardized value) indicates the number of standard deviations a given value x is above or below the mean.*

$$z = \frac{x-\mu}{\sigma}$$

A positive $z-$score means the data value is situated above the mean.

A zero $z-$score means the data value is situated exactly at the mean.

A negative $z-$score means the data value is situated below the mean.

Example 2. Find Suzanne's $z-$score on the verbal section. Recall, her raw score was 380, the mean was 356, and the standard deviation was 12.

$z_{\text{verbal}} = \frac{x-\mu}{\sigma}$	$z-$score formula Insert the score (x) and the mean (μ) for the verbal section
$z_{\text{verbal}} = \frac{380-356}{12}$	In the denominator insert the standard deviation (σ)
$z_{\text{verbal}} = \frac{24}{12}$	Divide 24 by 12
$z_{\text{verbal}} = 2$	This is her $z-$score for the verbal section.

Interpretation: Her verbal score was 2 standard deviations above the mean.

Example 3. Find Suzanne's $z-$score on the analytical section. Recall, her raw score was 420, the mean was 400, and the standard deviation was 20.

$z_{\text{analytical}} = \frac{x-\mu}{\sigma}$	$z-$score formula Insert the score (x) and the mean (μ) for the analytical section
$z_{\text{analytical}} = \frac{420-400}{20}$	In the denominator insert the standard deviation (σ)
$z_{\text{analytical}} = \frac{20}{20}$	Divide 20 by 20
$z_{\text{analytical}} = 1$	This is her $z-$score for the analytical section.

Interpretation: Her analytical score was 1 standard deviation above the mean.

Conclusion: Suzanne is certainly above average on both sections. We can quickly see that with the positive z-scores.

Example 4. On which section do you think she performed better?

The $z-$scores show that Suzanne's score on the verbal section is 2 standard deviations above the mean while her score on the analytical section is 1 standard deviation above the mean. Therefore, Suzanne performed better on the verbal section because it has a larger z-score.

Example 5. Computing a z-score

Liv's web page viewing varies from day to day with mean 200 pages and standard deviation 85 pages. Yesterday Liv viewed 305 web pages. What is her z-score?

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{305\ pages - 200\ pages}{85\ pages}$$

$$z = \frac{105\ \cancel{pages}}{85\ \cancel{pages}}$$

$$z = 1.24$$

Example 6. Using z-scores to compare groups

Angie (10 yr old) and Beth (17 yr old) are sisters and wanted to know who had the fastest time for the 50 meter free-style when compared to their own team.

	Swimmer Times	Team mean	Team standard deviation
Angie	34.6 sec	37.3 sec	2.0 sec
Beth	27.3 sec	30.1 sec	1.4 sec

$$z_{\text{Angie}} = \frac{x - \mu}{\sigma}$$

$$z_{\text{Angie}} = \frac{34.6sec - 37.3sec}{2.0sec}$$

$$z_{\text{Angie}} = \frac{-2.7\cancel{sec}}{2.0\cancel{sec}}$$

$$z_{\text{Angie}} = -1.35$$

$$z_{\text{Beth}} = \frac{x - \mu}{\sigma}$$

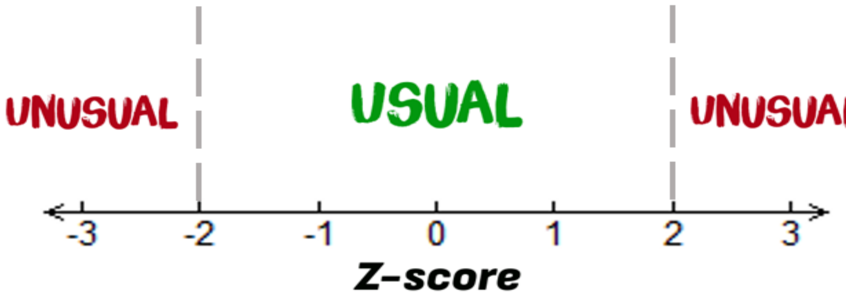
$$z_{\text{Beth}} = \frac{27.3sec - 30.1sec}{1.4sec}$$

$$z_{\text{Beth}} = \frac{-2.8\cancel{sec}}{1.4\cancel{sec}}$$

$$z_{\text{Beth}} = -2.8$$

Both girls had great swims that are less than their team's average. Once we standardize their times using their whole team's results, we can see that Beth had a faster swim time. Her z-score of -2.8 is farther from 0 than Angie's z-score of -1.35.

Objective: Unusual z-score



Remember, z-scores measure the number of standard deviations your data value is from the mean. A rule of thumb that we use in statistics is demonstrated in the diagram above.

- If a z-score is greater than 2 it is considered unusual. (Unusually high)
- If a z-score is less than -2 it is considered unusual. (Unusually low)
- If a z-score is between -2 and 2 it is considered usual.

Example 7. Determining if a data value is unusual using a z-score

A light bulb company claims that the average lifetime of its bulbs is 1000 hours with a standard deviation of 75 hours. If your light bulb only lasts 775 hours, is that an unusually short time to last, relatively speaking?

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{775\ hours - 1000\ hours}{75\ hours}$$

$$z = \frac{-225\ \cancel{hours}}{75\ \cancel{hours}}$$

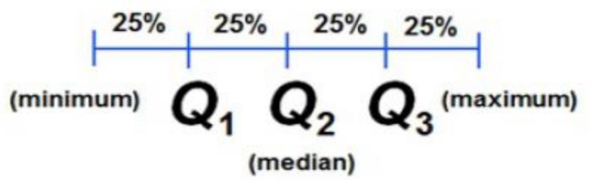
$$z = -3$$

Yes, because the z-score is -3.

Objective: Determine and interpret quartiles.

Quartiles are also measures of position. For example, if your child's pediatrician tells you that your child's height is at the third quartile (Q_3) for his/her age this means 75 percent of children of the same age as your child are of the same height or shorter than your child.

Definition 6. *As the name implies, **quartiles** of a data set divide the data set into four equal parts, each containing 25% of the data.*



- *The first quartile (Q_1) separates the bottom 25% of sorted values from the top 75%.*
- *The second quartile (Q_2) which is also the median separates the bottom 50% of sorted values from the top 50%.*
- *The third quartile (Q_3) separates the bottom 75% of sorted values from the top 25%.*

Steps to find quartiles:

1. Arrange the data in order of lowest to highest.
2. Find the median which is also the second quartile (Q_2)
3. Find the middle of the first half of the data which is the first quartile (Q_1)
4. Find the middle of the second half of the data which is the third quartile (Q_3)

Example 8. Commute distance to school

The following data is the commute distance (in miles) of a group of students. Find the quartiles for this data.

1	2	4	5	6	4	1	12	10	30	1	6
10	6	5	9	7	5	8	17	25	35	12	10

First, we order the values:

1 1 1 2 4 4 5 5 5 6 6 6 7 8 9 10 10 10 12 12 17 25 30 35

Second, we find the median (Q_2):

$$\text{Position of the median } = \frac{24+1}{2} = 12.5$$

We will need to determine what number is exactly between the 12th and 13th data value. The midpoint formula will help up determine this.

$$Q_2 = \text{Median} = \frac{6+7}{2} = 6.5$$

1 1 1 2 4 4 5 5 5 6 6 6 7 8 9 10 10 10 12 12 17 25 30 35

$Q_2 = 6.5$

Next, we find the middle of the first half of the data which is the first quartile (Q_1) and the middle of the second half of the data which is Q_3 (see picture below). Each of the halves have 12 data values so the middle is between each of their 6th and 7th data values.

first half of the data *second half of the data*

1 1 1 2 4 4 5 5 5 6 6 6 | 7 8 9 10 10 10 12 12 17 25 30 35

$Q_1 = 4.5$ $Q_3 = 11$

Thus, the quartiles for the data in this example are 4.5, 6.5, and 11, respectively.

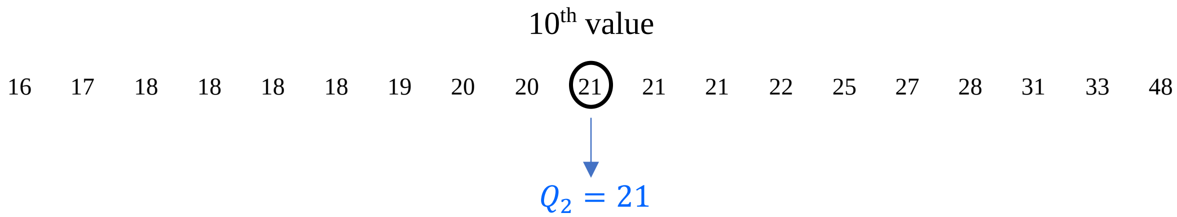
Example 9. Ages of students in an Algebra class.

The following data represents the ages of 19 students taking a winter session of *Intermediate Algebra*. **Note**: These ages are already sorted.

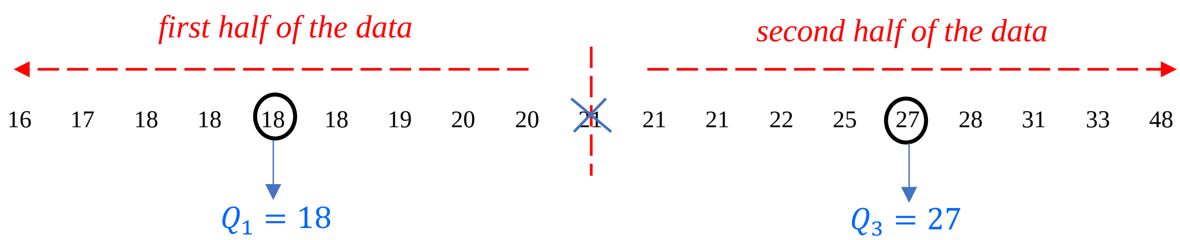
16 17 18 18 18 18 19 20 20 21 21 21 22 25 27 28 31 33 48

Find the median (Q_2):

Position of the median $\frac{19+1}{2} = 10^{th}$



Find Q_1(the middle of the first half of the data) and Q_3 (the middle of the second half of the data.



Thus, the quartiles for the data in this example are 18, 21, and 27, respectively.

Objective: Compute the five – number summary

Definition 11. ***A five number summary*** *consists of:*

i. *The minimum (smallest observation)*
ii. *The first quartile ($Q1$)*
iii. *The median ($Q2$)*
iv. *The third quartile ($Q3$)*
v. *The maximum (largest observation)*

Example 10. Write the 5 – number summary for the commute distance to school example.

$$Minimum = 1 \quad Q_1 = 4.5 \quad Q_2 = 6.5 \quad Q_3 = 11 \quad Maximum = 35$$

Example 11. Write the 5 – number summary for the ages of the algebra students example

$$Minimum = 16 \quad Q_1 = 18 \quad Q_2 = 21 \quad Q_3 = 27 \quad Maximum = 48$$

6.3 Practice

1. Suppose the average score on the commercial driver's license test in Maryland is 79 with a standard deviation of 5. Find the corresponding $z-$scores for each raw score.

   a. 89 b. 79 c. 77 d. 92 e. 81

2. The average cost in dollars of a wedding in United States (not including the cost for a honeymoon) is $26,650 with a standard deviation of $2,650. Find the corresponding $z-$scores for each raw score.

   a. $29,300 b. $10,750 c. $32,745

3. The z-score for the life span of your Apple iPhone is 0.78. Is your phone performing better than average or below average?

4. Among males, Martin's weight has a *z*-score of -1.90. Is Martin heavier or lighter than most men?

5. Suppose that the number of complaints in a month against police department B has a mean of 14 with a standard deviation of 2. This past month they had 11 complaints. Is this unusual relative to their other months?

6. After two years of school, a student who attends a university has a loan of $19,000 where the average debt is 16,000 and the standard deviation 1,500. After two years of school, another student who attends a college has a loan of 6,250 where the average debt is 6,000 and the standard deviation 500. Which student has a higher debt in relationship to his or her peers?

7. The tallest living man has a height of 243 cm. The tallest living woman is 234 cm tall. Heights of men have a mean of 173 cm and a standard deviation of 7 cm. Heights of women have a mean of 162 cm and a standard deviation of 5 cm. Relative to the population of the same gender, find who is taller.

8. Data from the last ten years shows that the average high school male has a personal best long jump of 18.9 feet with a standard deviation of 1.9 feet. The average high school female has a personal best jump of 16.5 feet with a standard deviation of 2.2 feet. Jason's longest jump is 20.2 feet. Sara's longest jump is 19 feet. Which athlete has a more extraordinary jump?

9. Juneau Alaska has an average snowfall in January of 24 inches with a standard deviation of 3.5 inches. Milwaukee Wisconsin has an average snowfall in January of 15 inches with a standard deviation of 2.8 inches. In January 2020, Juneau had 29.2 inches of snow and Milwaukee had 16.2 inches of snow. Which city is having a more extraordinary snowfall in January?

10. ***Multiple Choice***. Leon has a z-score of -1.3 for his Test 1 grade. Which of the following correctly interprets his z-score?
    a. Leon scored 1.3 points lower than the mean.
    b. Leon scored 1.3 points above the mean.
    c. Leon scored 1.3 standard deviations below the mean.
    d. Leon performed 1.3% lower than the class.

11. A group of 14 diners was asked how much they would pay (in dollars) for a meal. Their responses were (in dollars):

5	7	6	8	11	9	25	12	9	16	15	10	13	8

    Identify the five number summary for this data.

12. The data below are the number of wins in the regular season games for the Baltimore Ravens from 2004 to 2019.

10	6	13	5	11	9	12	12	10	8	10	5	8	9	10	14

    Identify the five number summary for this data.

13. The data below represents the number of wins in the 2019 regular season for 15 Major League Baseball teams. Identify the five number summary for this data.

Team	**Number of wins**
New York Yankees	103
Tampa Bay Rays	96
Boston Red Sox	84
Toronto Blue Jays	67
Baltimore Orioles	54
Minnesota Twins	101
Cleveland Indians	93
Chicago White Sox	72
Kansas City Royals	59
Detroit Tigers	47
Houston Astros	107
Oakland Athletics	97
Texas Rangers	78
Los Angeles Angels	72
Seattle Mariners	68

14. The following data is the pulse rates of 17 thirty year old females. Identify the five number summary for this data.

82	85	60	69	57	79	91	63	77	70	64	86	88	72	58	76	69

6.3 Answers

1. a. $z = 2$
   b. $z = 0$
   c. $z = -0.4$
   d. $z = 2.6$
   e. $z = 0.4$
2. a. $z = 1$
   b. $z = -6$
   c. $z = 2.30$
3. Above average because the z-score is positive.
4. Lighter than most men because his z-score is negative.

5. $z = \frac{11-14}{2} = -1.5$
   No, it is not unusually low because the z-score is not less than -2.

6. $z = 2$ for the student who attends the university
   $z = 0.5$ for the student who attends the college
   The student attending the university has a higher debt in relationship to his or her peers. This is because his or her z-score is larger.
7. $z_{\text{man}} = 10$
   $z_{\text{woman}} = 14.4$
   The woman is taller relative to the population of the same gender. This is because her $z-$score is larger.

8.

$$z_{Jason} = \frac{20.2 - 18.9}{1.9} = 0.68$$

$$z_{Sara} = \frac{19 - 16.5}{2.2} = 1.14$$

Sarah has a more extraordinary jump because her z-score is larger.

9.

$$z_{Juneau} = \frac{29.2 - 25}{3.4} = 1.2$$

$$z_{Milwaukee} = \frac{16.2 - 15}{2.8} = 0.43$$

Juneau had a more extraordinary snowfall in January 2020 because its z-score is larger.

10. c. Leon scored 1.3 standard deviations below the mean.

11. $Minimum = \$5 \quad Q_1 = \$8 \quad Median = \$9.50 \quad Q_3 = \$13 \quad Maximum = \$25$

12. $Minimum = 5 \quad Q_1 = 8 \quad Median = 10 \quad Q_3 = 11.5 \quad Maximum = 14$

13. $Minimum = 47\ wins,\ Q_1 = 67\, wins,\ Median = 78\, wins,\ Q_3 = 97\, wins,$ $Maximum = 107\, wins$

14. $Minimum = 57 \quad Q_1 = 63.5 \quad Median = 72 \quad Q_3 = 83.5 \quad Maximum = 91$