CHAPTER 1

Factoring Polynomials

Objectives
Chapter 1

- Find the greatest common factor of a polynomial.
- Factor the greatest common factor from a polynomial.
- Factor polynomials with four terms by grouping.
- Factor trinomials when the leading coefficient is 1.
- Factor trinomials using the ac method when the leading coefficient of the polynomial is not 1.
- Identify and factor special products including a difference of two perfect squares, perfect square trinomials, and sum and difference of two perfect cubes.
- Identify and use the correct method to factor various polynomials.
- Solve equations by factoring and using the zero product rule.

Section 1.1: Greatest Common Factor

Objectives: Find the greatest common factor of a polynomial. Factor the GCF from a polynomial.

The inverse of multiplying polynomials together is factoring polynomials. There are many benefits of factoring a polynomial. We use factored polynomials to help us solve equations, study behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials, it is very important to have strong factoring skills.

In this lesson, we will focus on factoring using the Greatest Common Factor or GCF of a polynomial. When multiplying monomials by polynomials, such as $4x^2(2x^2-3x+8)$, we distribute to get a product of $8x^4-12x^3+32x^2$. In this lesson, we will work backwards, starting with $8x^4-12x^3+32x^2$ and factoring to write as the product $4x^2(2x^2-3x+8)$.

DETERMINING THE GREATEST COMMON FACTOR

We will first introduce this idea by finding the GCF of several numbers. To find a GCF of several numbers, we look for the largest number that can divide each number without leaving a remainder.

Example 1. Determine the GCF of 15, 24, and 27.

$\frac{15}{3}=5, \frac{24}{3}=8, \frac{27}{3}=9$ Each of the numbers can be divided by 3

GCF = 3 Our Answer

When there are variables in our problem, we can first find the GCF of the numbers as in Example 1 above. Then we take any variables that are in common to all terms. The variable part of the GCF uses the *smallest* power of each variable that appears in all terms. This idea is shown in the next example.

Example 2. Determine the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$.

$\frac{24}{6}=4, \frac{18}{6}=3, \frac{12}{6}=2$ Each number can be divided by 6

Use the lowest exponent for each common variable; each term contains x^2y.
Note that z is not part of the GCF because the term $18x^2y^4$ does not contain the variable z.

GCF = $6x^2y$ Our Answer

FACTORING THE GREATEST COMMON FACTOR

Now we will learn to factor the GCF from a polynomial with two or more terms. Remember that factoring is the inverse process of multiplying. In particular, factoring the GCF reverses the distributive property of multiplication.

To factor the GCF from a polynomial, we first identify the GCF of all the terms. The GCF is the factor that goes in front of the parentheses. Then we divide each term of the given polynomial by the GCF. For the second factor, enclose the quotients within the parentheses. In the final answer, the GCF is outside the parentheses and the remaining quotients are enclosed within the parentheses.

Example 3. Factor using the GCF.

$4x^2 - 20x - 16$ GCF of $4x^2$, $-20x$, and 16 is 4; divide each term by 4

$$\frac{4x^2}{4} = x^2, \frac{-20x}{4} = -5x, \frac{-16}{4} = -4$$

The quotients are the terms left inside the parentheses; keep the GCF outside the parentheses

$= 4(x^2 - 5x - 4)$ Our Answer

With factoring, we can always check our answers by multiplying (distributing); the resulting product should be the original expression.

Example 4. Factor using the GCF.

$25x^4 - 15x^3 + 20x^2$ GCF of $25x^4$, $-15x^3$, and $20x^2$ is $5x^2$; divide each term by $5x^2$

$$\frac{25x^4}{5x^2} = 5x^2, \frac{-15x^3}{5x^2} = -3x, \frac{20x^2}{5x^2} = 4$$

These quotients are the terms left inside the parentheses; keep the GCF outside the parentheses

$= 5x^2(5x^2 - 3x + 4)$ Our Answer

Example 5. Factor using the GCF.

$3x^3y^2z + 5x^4y^3z^5 - 4xy^4$ GCF of $3x^3y^2z$, $5x^4y^3z^5$, and $-4xy^4$ is xy^2; divide each term by xy^2

$$\frac{3x^3y^2z}{xy^2} = 3x^2z, \frac{5x^4y^3z^5}{xy^2} = 5x^3yz^5, \frac{-4xy^4}{xy^2} = -4y^2$$

These quotients are the terms left inside the parentheses; keep the GCF outside the parentheses

$= xy^2(3x^2z + 5x^3yz^5 - 4y^2)$ Our Answer

Example 6. Factor using the GCF.

$21x^3 + 14x^2 + 7x$ GCF of $21x^3$, $14x^2$, and $7x$ is $7x$;
divide each term by $7x$

$$\frac{21x^3}{7x} = 3x^2, \frac{14x^2}{7x} = 2x, \frac{7x}{7x} = 1$$

The factors are the GCF and the result of the division;
These quotients are the terms left inside the parentheses;
keep the GCF outside the parentheses.

$= 7x(3x^2 + 2x + 1)$ Our Answer

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was also one of the terms, so dividing resulting in a quotient of 1. Factoring will never make terms disappear. Anything divided by itself is 1; be sure not to forget to put the 1 into the solution.

Often the second line is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parentheses containing the remaining factors as shown in the following example.

Example 7. Factor using the GCF.

$18a^4b^3 - 27a^3b^3 + 9a^2b^3$ GCF is $9a^2b^3$, divide each term by $9a^2b^3$

$= 9a^2b^3(2a^2 - 3a + 1)$ Our Answer

Again, in the previous problem when dividing $9a^2b^3$ by itself, the result is 1. Be very careful that each term is accounted for in your final solution.

GREATEST COMMON FACTOR EQUAL TO 1

Sometimes an expression has a GCF of 1. If there is no common factor other than 1, the polynomial expression cannot be factored using the GCF. This is shown in the following example.

Example 8. Factor using the GCF.

$8ab - 17c + 49$ GCF is 1 because there are no other factors in common to all 3 terms

cannot be factored using the GCF Our Answer

FACTORING THE *NEGATIVE* OF THE GCF

If the first term of a polynomial has a negative coefficient, always make the GCF negative in order to make the first term inside the parentheses have a positive coefficient. See Example 9 on the next page.

Example 9. Factor using the GCF.

$-12x^5y^2+6x^4y^4-8x^3y^5$ GCF of $\left(-12x^5y^2\right)$, $\left(6x^4y^4\right)$, and $\left(-8x^3y^5\right)$ is $(-2x^3y^2)$;
because the first term is negative;
divide each term by $(-2x^3y^2)$

$$\frac{-12x^5y^2}{-2x^3y^2}=6x^2,\ \frac{6x^4y^4}{-2x^3y^2}=-3xy^2,\ \frac{-8x^3y^5}{-2x^3y^2}=4y^3$$

The results are what is left inside the parentheses

$=-2x^3y^2(6x^2-3xy^2+4y^3)$ Our Answer

We will always begin factoring by looking for a Greatest Common Factor and factoring it out if there is one. In the rest of this chapter, we will learn other factoring techniques that might be used to write a polynomial as a product of prime polynomials.

Practice Exercises
Section 1.1: Greatest Common Factor

Factor using the GCF.
If the GCF is 1, state that the polynomial "cannot be factored using the GCF".

1) $15x+20$

2) $12-8x$

3) $9x-9$

4) $3x^2+5x$

5) $10x^3-18x$

6) $7ab-35a^2b$

7) $9+8x^2$

8) $4x^3y^2+8x^3$

9) $24x^2y^5-18x^3y^2$

10) $-3a^2b+6a^3b^2$

11) $5x^3-7$

12) $-32n^9+32n^6+40n^5$

13) $20x^4-30x+30$

14) $21p^6+30p^2+27$

15) $28m^4+40m^3+8$

16) $-10x^4+20x^2+12x$

17) $30b^9+5ab-15a^2$

18) $27y^7+12xy^2+9y^2$

19) $-48a^2b^2-56a^3b-56a^5b$

20) $30m^6+15mn^2-25$

21) $20x^8y^2z^2+15x^5y^2z+35x^3y^3z$

22) $3p+12q-15q^2r^2$

23) $50x^2y+10y^2+70xz^2$

24) $30x^5y^4z^3+50y^4z^5-10xy^4z^3$

25) $30pqr-5pq+5q$

26) $28b+14b^2+35b^3+7b^5$

27) $-18n^5+3n^3-21n+3$

28) $30a^8+6a^5+27a^3+21a^2$

29) $-40x^{11}-20x^{12}+50x^{13}-50x^{14}$

30) $-24x^6-4x^4+12x^3+4x^2$

31) $-32mn^8+4m^6n+12mn^4+16mn$

32) $-10y^7+6y^{10}-4xy^{10}-8xy^8$

ANSWERS to Practice Exercises
Section 1.1: Greatest Common Factor

1) $5(3x+4)$
2) $4(3-2x)$
3) $9(x-1)$
4) $x(3x+5)$
5) $2x(5x^2-9)$
6) $7ab(1-5a)$
7) cannot be factored using the GCF
8) $4x^3(y^2+2)$
9) $6x^2y^2(4y^3-3x)$
10) $-3a^2b(1-2ab)$
11) cannot be factored using the GCF
12) $-8n^5(4n^4-4n-5)$
13) $10(2x^4-3x+3)$
14) $3(7p^6+10p^2+9)$
15) $4(7m^4+10m^3+2)$
16) $-2x(5x^3-10x-6)$
17) $5(6b^9+ab-3a^2)$
18) $3y^2(9y^5+4x+3)$
19) $-8a^2b(6b+7a+7a^3)$
20) $5(6m^6+3mn^2-5)$
21) $5x^3y^2z(4x^5z+3x^2+7y)$
22) $3(p+4q-5q^2r^2)$
23) $10(5x^2y+y^2+7xz^2)$
24) $10y^4z^3(3x^5+5z^2-x)$
25) $5q(6pr-p+1)$
26) $7b(4+2b+5b^2+b^4)$
27) $-3(6n^5-n^3+7n-1)$
28) $3a^2(10a^6+2a^3+9a+7)$
29) $-10x^{11}(4+2x-5x^2+5x^3)$
30) $-4x^2(6x^4+x^2-3x-1)$
31) $-4mn(8n^7-m^5-3n^3-4)$
32) $-2y^7(5-3y^3+2xy^3+4xy)$

Section 1.2: Factoring by Grouping

Objective: Factor polynomials with four terms by grouping.

Whenever possible, we will always do when factoring a polynomial is factor out the greatest common factor (GCF). This GCF is often a monomial. For example, the GCF of $5xy+10xz$ is the monomial $5x$, so we would factor as $5x(y+2z)$. However, a GCF does not have to be a monomial; it could be a polynomial. To see this, consider the following two examples.

Example 1. Factor completely.

$3ax-7bx$	Both terms have x in common, factor it out
$=x(3a-7b)$	Our Answer

Now we have a similar problem, but instead of the monomial x, we have the binomial $(2a+5b)$ as the GCF.

Example 2. Factor completely.

$3a(2a+5b)-7b(2a+5b)$	Both terms have $(2a+5b)$ in common, factor it out
$=(2a+5b)(3a-7b)$	Our Answer

In the same way we factored out the GCF of x, we can factor out the GCF which is a binomial, $(2a+5b)$.

FACTORING BY GROUPING

When a polynomial has a GCF of 1, it still may be factorable. Additional factoring strategies will be needed.

When a polynomial has **four** terms, we will attempt to factor it using a strategy called **grouping**.

Remember, factoring is the reverse of multiplying, so first we will look at a multiplication problem and then try to reverse the process.

Example 3. Multiply.

$(2a+3)(5b+2)$	Distribute $(2a+3)$ to each term in the second parentheses
$=5b(2a+3)+2(2a+3)$	Distribute each monomial
$=10ab+15b+4a+6$	Our Answer

The product has four terms. We arrived at this answer by looking at the two parts, $5b(2a+3)$ and $2(2a+3)$.

When we are factoring by grouping, we split the expression into two groups: the first two terms and the last two terms. Then we can factor the GCF out of each group of two terms. When we do this, our hope is what remains in the parentheses will match in both the left group and the right group. If they match, we can pull this matching binomial GCF out front, putting the rest in parentheses and the expression will be factored.

The next example is the same problem worked backwards, factoring instead of multiplying.

Example 4. Factor completely.

$10ab+15b+4a+6$	Split expression into two groups
$= \boxed{10ab+15b}\boxed{+4a+6}$	Factor the GCF from each group of two terms
$= \boxed{5b(2a+3)}\boxed{+2(2a+3)}$	$(2a+3)$ is common to both terms; Factor out this binomial GCF
$= (2a+3)(5b+2)$	Our Answer

Example 5. Factor completely.

$6x^3-15x^2+2x-5$	Split expression into two groups
$= \boxed{6x^3-15x^2}\boxed{+2x-5}$	Factor the GCF from each group of two terms
$= \boxed{3x^2(2x-5)}\boxed{+1(2x-5)}$	$(2x-5)$ is common to both terms; Factor out this binomial GCF
$= (2x-5)(3x^2+1)$	Our Answer

The key for grouping to work is after the GCF is factored out of the left and right groups, the two binomials must match exactly. If there is any difference between the two binomials, we either have to do some adjusting or it can't be factored using the grouping method.

Sometimes, we will need to factor the negative of the GCF of a grouping to be sure the remaining binomials match.

Example 6. Factor completely.

$6x^2+9xy-14x-21y$	Split expression into two groups
$= \boxed{6x^2+9xy}\boxed{-14x-21y}$	Factor the GCF from each group of two terms
$= \boxed{3x(2x+3y)}\boxed{-7(2x+3y)}$	$(2x+3y)$ is common to both terms; Factor out this binomial GCF
$= (2x+3y)(3x-7)$	Our Answer

Example 7. Factor completely.

$5xy-8x-10y+16$	Split expression into two groups
$=\boxed{5xy-8x}\boxed{-10y+16}$	Factor the GCF from each group of two terms
$=\boxed{x(5y-8)}\boxed{-2(5y-8)}$	$(5y-8)$ is common to both terms; Factor out this binomial GCF
$=(5y-8)(x-2)$	Our Answer

Example 8. Factor completely.

$12ab-14a-6b+7$	Split expression into two groups
$=\boxed{12ab-14a}\boxed{-6b+7}$	Factor the GCF from each group of two terms
$=\boxed{2a(6b-7)}\boxed{-1(6b-7)}$	$(6b-7)$ is common to both terms; Factor out this binomial GCF
$=(6b-7)(2a-1)$	Our Answer

Example 9. Factor completely.

$4a^2+6ab-14ab^2-21b^3$	Split expression into two groups
$=\boxed{4a^2+6ab}\boxed{-14ab^2-21b^3}$	Factor the GCF from each group of two terms
$=\boxed{2a(2a+3b)}\boxed{-7b^2(2a+3b)}$	$(2a+3b)$ is common to both terms; Factor out this binomial GCF
$=(2a+3b)(2a-7b^2)$	Our Answer

Example 10. Factor completely.

$8xy-12y-10x+15$	Split expression into two groups
$=\boxed{8xy-12y}\boxed{-10x+15}$	Factor the GCF from each group of two terms
$=\boxed{4y(2x-3)}\boxed{-5(2x-3)}$	$(2x-3)$ is common to both terms; Factor out this binomial GCF
$=(2x-3)(4y-5)$	Our Answer

Sometimes the terms in the expression must be rearranged in order for factoring by grouping to work.

Example 11. Factor completely.

$6xy+4+3x+8y$	Split expression into two groups
$= \boxed{6xy+4} \boxed{+3x+8y}$	Factor the GCF from each group of two terms
$= \boxed{2(3xy+2)} \boxed{+1\,(3x+8y)}$	The remaining factors are not the same; rearrange the terms.
$6xy+3x+8y+4$	Split expression into two groups
$= \boxed{6xy+3x} \boxed{+8y+4}$	Factor the GCF from each group of two terms
$= \boxed{3x(2y+1)} \boxed{+4\,(2y+1)}$	$(2y+1)$ is common to both terms; Factor out this binomial GCF
$= (2y+1)\,(3x+4)$	Our Answer

Practice Exercises
Section 1.2: Factoring by Grouping

Factor completely.

1) $x^3+3x^2+4x+12$
2) $x^3-3x^2+6x-18$
3) $x^3-5x^2-2x+10$
4) x^3+x^2-3x-3
5) $40r^3-8r^2-25r+5$
6) $35x^3-10x^2-56x+16$
7) $3n^3-2n^2-9n+6$
8) $14v^3+10v^2-7v-5$
9) $15b^3+21b^2-35b-49$
10) $6x^3-48x^2+5x-40$
11) $3x^3+15x^2+2x+10$
12) $35x^3-28x^2-20x+16$
13) $7n^3+21n^2-5n-15$
14) $7xy-49x+5y-35$
15) $42r^3-49r^2+18r-21$
16) $32xy+40x^2+12y+15x$
17) $15ab-6a+5b^3-2b^2$
18) $16xy-56x+2y-7$
19) $3mn-8m+15n-40$
20) $5mn+2m-25n-10$
21) $40xy+35x-8y^2-7y$
22) $32uv-20u+24v-15$
23) $10xy+30+25x+12y$
24) $24xy+25y^2-20x-30y^3$
25) $3uv+14u-6u^2-7v$
26) $56ab+14-49a-16b$
27) $2xy-8x^2+7y^3-28xy^2$
28) $28p^3+21p^2+20p+15$
29) $16xy-3x-6x^2+8y$
30) $8xy+56x-y-7$

ANSWERS to Practice Exercises
Section 1.2: Factoring by Grouping

1) $(x+3)(x^2+4)$

2) $(x-3)(x^2+6)$

3) $(x-5)(x^2-2)$

4) $(x+1)(x^2-3)$

5) $(5r-1)(8r^2-5)$

6) $(7x-2)(5x^2-8)$

7) $(3n-2)(n^2-3)$

8) $(7v+5)(2v^2-1)$

9) $(5b+7)(3b^2-7)$

10) $(x-8)(6x^2+5)$

11) $(x+5)(3x^2+2)$

12) $(5x-4)(7x^2-4)$

13) $(n+3)(7n^2-5)$

14) $(y-7)(7x+5)$

15) $(6r-7)(7r^2+3)$

16) $(4y+5x)(8x+3)$

17) $(5b-2)(3a+b^2)$

18) $(2y-7)(8x+1)$

19) $(3n-8)(m+5)$

20) $(5n+2)(m-5)$

21) $(8y+7)(5x-y)$

22) $(8v-5)(4u+3)$

23) $(2y+5)(5x+6)$

24) $(6y-5)(4x-5y^2)$

25) $(v-2u)(3u-7)$

26) $(8b-7)(7a-2)$

27) $(y-4x)(2x+7y^2)$

28) $(4p+3)(7p^2+5)$

29) $(8y-3x)(2x+1)$

30) $(y+7)(8x-1)$

Section 1.3: Factor Trinomials Whose Leading Coefficient is 1

Objective: Factor trinomials when the leading coefficient is 1.

We will now learn a strategy for factoring trinomials (polynomials with three terms). In this section, we will focus on trinomials of the form $x^2 + bx + c$. In this case, the leading coefficient (the coefficient of the first term) is 1.

Since factoring is the reverse of multiplication, we will start with a multiplication problem and look at how we can reverse the process.

Example 1. Multiply and simplify.

$(x+6)(x-4)$	Distribute $(x+6)$ to each term in the second set of parentheses
$= x(x+6)-4(x+6)$	Distribute each monomial through each se of parentheses
$= x^2+6x-4x-24$	Combine like terms
$= x^2+2x-24$	Our Answer

Notice that if you reverse the last three steps, the process is factoring by grouping! The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This process is shown in the following example, which is this same problem worked backwards:

Example 2. Factor completely.

$x^2+2x-24$	Replace the middle term $+2x$ with $+6x-4x$
$= x^2+6x-4x-24$	Split expression into two pairs of terms; factor the GCF from each pair
$= x(x+6)-4(x+6)$	$(x+6)$ is common to both terms*:* factor this binomial GCF
$= (x+6)(x-4)$	Our Answer

The key to making these problems work is in the way we split the middle term. Why did we choose $+6x-4x$ and not $+5x-3x$? To find the correct way to split the middle term, we will use what is called the *ac method*. The *ac method* works by finding a pair of numbers that multiply to obtain the last number in the trinomial **and** also add up to the coefficient of the middle term of the trinomial. In the previous example, the numbers must multiply to -24 and add to $+2$. The only numbers that can do this are 6 and -4. Notice that $6\cdot(-4) = -24$ and $6+(-4) = 2$.

FACTORING TRINOMIALS OF THE FORM $x^2 + bx + c$

Example 3. Factor completely.

$x^2 + 9x + 18$	Find factors that multiply to 18 and add to 9: Use 6 and 3 Replace $9x$ with $6x + 3x$
$= x^2 + 6x + 3x + 18$	Factor by grouping
$= x(x+6) + 3(x+6)$	
$= (x+6)(x+3)$	Our Answer

Example 4. Factor completely.

$x^2 - 4x + 3$	Find factors that multiply to 3 and add to -4: Use -3 and -1 Replace $-4x$ with $-3x + (-1x) = -3x - 1x$
$= x^2 - 3x - x + 3$	Factor by grouping
$= x(x-3) - 1(x-3)$	
$= (x-3)(x-1)$	Our Answer

Example 5. Factor completely.

$x^2 - 8x - 20$	Find factors that multiply to -20 and add to -8: Use -10 and 2 Replace $-8x$ with $-10x + 2x$
$= x^2 - 10x + 2x - 20$	Factor by grouping
$= x(x-10) + 2(x-10)$	
$= (x-10)(x+2)$	Our Answer

FACTORING TRINOMIALS IN TWO VARIABLES

Often we are asked to factor trinomials in two variables. The *ac method* works in much the same way: Find a pair of terms that multiplies to obtain the last term in the trinomial **and** also adds up to the the middle term of the trinomial.

Example 6. Factor completely.

$a^2 - 9ab + 14b^2$	Find factors that multiply to 14 and add to -9: Use -7 and -2 Replace $-9ab$ with $-7ab - 2ab$
$= a^2 - 7ab - 2ab + 14b^2$	Factor by grouping
$= a(a-7b) - 2b(a-7b)$	
$= (a-7b)(a-2b)$	Our Answer

As the past few examples illustrate, it is very important to find the correct pair of terms we will use to replace the middle term. Consider the following example, done **incorrectly**, where we mistakenly found two factors that multiply to 6 instead of -6:

Warning!

x^2+5x-6	Find factors that multiply to 6 and add to 5:
	Use 2 and 3
	Replace $5x$ with $2x+3x$
$=x^2+2x+3x-6$	Factor by grouping
$=x(x+2)+3(x-2)$	
???	Binomials do not match!

Because we did not use the negative sign with the 6 to find our pair of terms, the binomials did not match and grouping was not able to work at the end. The problem is done correctly below by choosing the correct pair of terms:

Example 7. Factor completely.

x^2+5x-6	Find factors that multiply to -6 and add to 5:
	Use 6 and -1
	Replace $5x$ with $6x-1x$
$=x^2+6x-1x-6$	Factor by grouping
$=x(x+6)-1(x+6)$	
$=(x+6)(x-1)$	Our Answer

FACTORING SHORTCUT

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1; our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it, however, when the leading coefficient of the trinomial is 1. In all of the problems we have factored in this lesson, the leading coefficient is 1. If this is the case, then we can use this shortcut. This process is shown in the next few examples.

Example 8. Factor completely.

$x^2-7x-18$	Find factors that multiply to -18 and add to -7:
	Use -9 and 2
	Write the binomial factors using -9 and 2
$=(x-9)(x+2)$	Our Answer

Example 9. Factor completely.

$m^2 - mn - 30n^2$	Find factors that multiply to -30 and add to -1:
	Use 5 and -6
	Write the binomial factors using 5 and -6
	Don't forget the second variable
$= (m + 5n)(m - 6n)$	Our Answer

It is possible to have an expression that does not factor. If there is no combination of numbers that multiply and add up to the correct numbers, then we cannot factor the polynomial and we say the polynomial is *prime*. This is shown in the following example.

Example 10. Factor completely.

$x^2 + 2x + 6$	Find factors that multiply to 6 and add to 2:
	$1 \cdot 6$, $2 \cdot 3$, $(-1)\cdot(-6)$, and $(-2)\cdot(-3)$ are the only
	ways to multiply to 6 but none of these pairs adds to 2
prime	Our Answer

FACTORING USING MORE THAN ONE STRATEGY

When factoring any polynomial, it is important to first factor any GCF (other than 1). If all of the terms in a polynomial have a common factor other than 1, we will want to first factor out the GCF before attempting any other method. The next three examples illustrate this technique.

Example 11. Factor completely.

$3x^2 - 24x + 45$	GCF of all three terms is 3; divide each term by 3
$= 3(x^2 - 8x + 15)$	Find factors that multiply to 15 and add to -8:
	Use -5 and -3
	Write the binomial factors using -5 and -3
$= 3(x - 5)(x - 3)$	Our Answer

Example 12. Factor completely.

$4x^2y - 8xy - 32y$	GCF of all three terms is $4y$; divide each terms by $4y$
$= 4y(x^2 - 2x - 8)$	Find factors that multiply to -8 and add to -2:
	Use -4 and 2
	Write the binomial factors using -4 and 2
$= 4y(x - 4)(x + 2)$	Our Answer

Example 13. Factor completely.

$7a^4b^2+28a^3b^2-35a^2b^2$	GCF of all three terms is $7a^2b^2$*;* divide each term by $7a^2b^2$
$=7a^2b^2(a^2+4a-5)$	Find factors that multiply to -5 and to 4: Use -1 and 5
	Write the binomial factors using -1 and 5
$=7a^2b^2(a-1)(a+5)$	Our Answer

Again it is important to comment on the shortcut of jumping right to the factors. This shortcut only works if the leading coefficient is 1. In the next lesson, we will look at how this process changes slightly when we have a number other than 1 as the leading coefficient.

Practice Exercises
Section 1.3: Factoring Trinomials Whose Leading Coefficient is 1

Factor completely.

1) $x^2+12x+32$

2) $x^2+13x+40$

3) $x^2-7x+10$

4) x^2-9x+8

5) x^2+x-30

6) x^2+x-72

7) $x^2-6x-27$

8) $x^2-9x-10$

9) $y^2-17y+70$

10) $x^2+3x-18$

11) $p^2+17p+72$

12) $x^2+3x-70$

13) $p^2+15p+54$

14) $p^2+7p-30$

15) c^2-4c+9

16) $m^2-15mn+50n^2$

17) $u^2-10uv+21v^2$

18) $m^2-3mn-40n^2$

19) $m^2+2mn-8n^2$

20) $x^2+10xy+16y^2$

21) $x^2-11xy+18y^2$

22) $u^2-10uv-24v^2$

23) $x^2+xy-12y^2$

24) $x^2+14xy+45y^2$

25) $x^2+4xy-12y^2$

26) $4x^2+52x+168$

27) $5a^2+60a+100$

28) $7w^2+35w-63$

29) $6a^2+24a-192$

30) $-5v^2-20v+25$

31) $6x^2+18xy+12y^2$

32) $5m^2+30mn-80n^2$

33) $6x^2+96xy+378y^2$

34) $-n^2+15n-56$

35) $-16t^2+48t+64$

ANSWERS to Practice Exercises

Section 1.3: Factoring Trinomials Whose Leading Coefficient is 1

1) $(x+8)(x+4)$

2) $(x+8)(x+5)$

3) $(x-2)(x-5)$

4) $(x-1)(x-8)$

5) $(x-5)(x+6)$

6) $(x+9)(x-8)$

7) $(x-9)(x+3)$

8) $(x-10)(x+1)$

9) $(y-10)(y-7)$

10) $(x+6)(x-3)$

11) $(p+9)(p+8)$

12) $(x+10)(x-7)$

13) $(p+6)(p+9)$

14) $(p+10)(p-3)$

15) prime

16) $(m-5n)(m-10n)$

17) $(u-7v)(u-3v)$

18) $(m+5n)(m-8n)$

19) $(m+4n)(m-2n)$

20) $(x+8y)(x+2y)$

21) $(x-9y)(x-2y)$

22) $(u-12v)(u+2v)$

23) $(x-3y)(x+4y)$

24) $(x+5y)(x+9y)$

25) $(x+6y)(x-2y)$

26) $4(x+7)(x+6)$

27) $5(a+10)(a+2)$

28) $7(w^2+5w-9)$

29) $6(a-4)(a+8)$

30) $-5(v-1)(v+5)$

31) $6(x+2y)(x+y)$

32) $5(m-2n)(m+8n)$

33) $6(x+9y)(x+7y)$

34) $-(n-8)(n-7)$

35) $-16(t-4)(t+1)$

Section 1.4: Factor Trinomials Whose Leading Coefficient is not 1

Objective: Factor trinomials using the *ac* method when the leading coefficient of the polynomial is not 1.

When factoring trinomials, we use the *ac method* to split the middle term and then factor by grouping. The *ac method* gets its name from the general trinomial expression, ax^2+bx+c, where a, b, and c are the numbers (coefficients) in front of x^2 and x terms, and the constant at the end, respectively.

The *ac method* is named ac because we multiply $a\cdot c$ to find out what we want the two numbers to multiply to. In the previous lesson, we always multiplied to just c because there was no number written in front of x^2. This meant the leading coefficient was 1 and we were multiplying $1\cdot c$, which is c. Now we will have a number other than 1 as the leading coefficient; so, we will be looking for two numbers that multiply to $a\cdot c$ and that add to b.

FACTORING TRINOMIALS OF THE FORM ax^2+bx+c WHERE $a\neq 1$

Example 1. Factor completely.

$3x^2+11x+6$	Multiply $a\cdot c$: $(3)(6)=18$
	Find factors that multiply to 18 and add to 11:
	Use 9 and 2
	Replace $11x$ with $9x+2x$
$=3x^2+9x+2x+6$	Factor by grouping
$=3x(x+3)+2(x+3)$	
$=(x+3)(3x+2)$	Our Answer

When $a=1$, we are able to use the shortcut discussed in the previous section, using the numbers that split the middle term to write the binomial factors. Example 1 above illustrates that the shortcut does not work when $a\neq 1$. **We must go through *all* the steps of grouping in order to factor the expression.**

Example 2. Factor completely.

$8x^2-2x-15$	Multiply $a\cdot c$: $(8)(-15)=-120$
	Find factors that multiply to -120 and add to -2:
	Use -12 and 10
	Replace $-2x$ with $-12x+10x$
$=8x^2-12x+10x-15$	Factor by grouping
$=4x(2x-3)+5(2x-3)$	
$=(2x-3)(4x+5)$	Our Answer

Example 3. Factor completely.

$10x^2 - 27x + 5$	Multiply $a \cdot c$: $(10)(5) = 50$ Find factors that multiply to 50 and add to -27: Use -25 and -2 Replace $-27x$ with $-25x - 2x$
$= 10x^2 - 25x - 2x + 5$	Factor by grouping
$= 5x(2x-5) - 1(2x-5)$	
$= (2x-5)(5x-1)$	Our Answer

The same process works with polynomials in two variables.

Example 4. Factor completely.

$4x^2 - xy - 5y^2$	Multiply $a \cdot c$: $(4)(-5) = -20$ Find factors that multiply to -20 and add to -1: Use 4 and -5 Replace $-xy$ with $4xy - 5xy$
$= 4x^2 + 4xy - 5xy - 5y^2$	Factor by grouping
$= 4x(x+y) - 5y(x+y)$	
$= (x+y)(4x-5y)$	Our Answer

FACTORING USING MORE THAN ONE STRATEGY

As always, when factoring we will first look for a GCF before using any other method, including the *ac method*. Factoring out the GCF first also has the added bonus of making the numbers smaller so the *ac method* becomes easier.

Example 5. Factor completely.

$18x^3 + 33x^2 - 30x$	$\text{GCF} = 3x$; factor from each term
$= 3x[6x^2 + 11x - 10]$	Multiply $a \cdot c$: $(6)(-10) = -60$ Find factors that multiply to -60 and add to 11: Use 15 and -4 Replace $11x$ with $15x - 4x$
$= 3x[6x^2 + 15x - 4x - 10]$	Factor by grouping
$= 3x[3x(2x+5) - 2(2x+5)]$	
$= 3x(2x+5)(3x-2)$	Our Answer

As was the case with trinomials when $a = 1$, not all trinomials can be factored. If there is no combination of numbers that multiplies and adds up to the correct numbers, then we cannot factor the polynomial and we say the polynomial is *prime*.

Example 6. Factor completely.

$3x^2 + 2x - 7$	Multiply $a \cdot c$: $(3)(-7) = -21$ $-3(7)$, $-7(3)$, $-1(21)$, and $-21(1)$ are the only ways to multiply to -21, but none of these pairs sums to 2
prime	Our Answer

Practice Exercises

Section 1.4: Factoring Trinomials Whose Leading Coefficient is not 1

Factor completely.

1) $5x^2+13x+6$
2) $2x^2-5x+2$
3) $3r^2-4r-4$
4) $4r^2+3r-7$
5) $2x^2-x+3$
6) $4k^2-17k+4$
7) $2b^2-b-3$
8) $6p^2+11p-7$
9) $4r^2+r-3$
10) $2x^2+19x+35$
11) $3x^2-17x+20$
12) $5n^2-4n-20$
13) $7x^2-48x+36$
14) $7n^2-44n+12$
15) $-7x^2-15x-2$
16) $7v^2-24v-16$
17) $5a^2-13a-28$
18) $7x^2+29x-30$
19) $5k^2-26k+24$
20) $-3r^2-16r-21$
21) $3u^2+13uv-10v^2$
22) $3x^2+17xy+10y^2$
23) $7x^2-2xy-5y^2$
24) $5x^2+28xy-49y^2$
25) $5u^2+31uv-28v^2$
26) $6x^2-39x-21$
27) $-10a^2+54a+36$
28) $21k^2-87k-90$
29) $-21n^2-45n+54$
30) $14x^2-60x+16$
31) $6x^2+29x+20$
32) $4x^2+9xy+2y^2$
33) $4m^2+6mn+6n^2$
34) $4m^2-9mn-9n^2$
35) $4x^2-6xy+30y^2$
36) $4x^2+13xy+3y^2$
37) $18u^2-3uv-36v^2$
38) $12x^2+62xy+70y^2$
39) $16x^2+60xy+36y^2$
40) $24x^2-52xy+8y^2$

ANSWERS to Practice Exercises
Section 1.4: Factoring Trinomials Whose Leading Coefficient is not 1

1) $(5x+3)(x+2)$
2) $(2x-1)(x-2)$
3) $(3r+2)(r-2)$
4) $(r-1)(4r+7)$
5) prime
6) $(k-4)(4k-1)$
7) $(2b-3)(b+1)$
8) $(3p+7)(2p-1)$
9) $(r+1)(4r-3)$
10) $(2x+5)(x+7)$
11) $(3x-5)(x-4)$
12) prime
13) $(7x-6)(x-6)$
14) $(7n-2)(n-6)$
15) $-(7x+1)(x+2)$
16) $(7v+4)(v-4)$
17) $(5a+7)(a-4)$
18) $(7x-6)(x+5)$
19) $(5k-6)(k-4)$
20) $-(3r+7)(r+3)$
21) $(3u-2v)(u+5v)$
22) $(3x+2y)(x+5y)$
23) $(7x+5y)(x-y)$
24) $(5x-7y)(x+7y)$
25) $(5u-4v)(u+7v)$
26) $3(2x+1)(x-7)$
27) $-2(5a+3)(a-6)$
28) $3(7k+6)(k-5)$
29) $-3(7n-6)(n+3)$
30) $2(7x-2)(x-4)$
31) $(x+4)(6x+5)$
32) $(x+2y)(4x+y)$
33) $2(2m^2+3mn+3n^2)$
34) $(m-3n)(4m+3n)$
35) $2(2x^2-3xy+15y^2)$
36) $(x+3y)(4x+y)$
37) $3(3u+4v)(2u-3v)$
38) $2(2x+7y)(3x+5y)$
39) $4(x+3y)(4x+3y)$
40) $4(x-2y)(6x-y)$

Section 1.5: Factoring Special Products

Objective: Identify and factor special products including a difference of two perfect squares, perfect square trinomials, and sum and difference of two perfect cubes.

When factoring there are a few special products that, if we can recognize them, help us factor polynomials.

DIFFERENCE OF TWO PERFECT SQUARES

When multiplying special products, we found that a sum of a binomial and a difference of a binomial could multiply to a difference of two perfect squares. Here, we will use this special product to help us factor.

Difference of Two Perfect Squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Example 1. Factor completely.

$x^2 - 16$	Express each term as the square of a monomial
$= (x)^2 - (4)^2$	Apply the difference of two perfect squares formula: Here, $a = x$ and $b = 4$
$= (x+4)(x-4)$	Our Answer

Example 2. Factor completely.

$36 - y^2$	Express each term as the square of a monomial
$= (6)^2 - (y)^2$	Apply the difference of two perfect squares formula: Here, $a = 6$ and $b = y$
$= (6+y)(6-y)$	Our Answer

Example 3. Factor completely.

$9a^2 - 25b^2$	Express each term as the square of a monomial
$= (3a)^2 - (5b)^2$	Apply the difference of two perfect squares formula: Here, $a = 3a$ and $b = 5b$
$= (3a+5b)(3a-5b)$	Our Answer

PERFECT SQUARE TRINOMIAL

Another special case involves the perfect square trinomial. We had a shortcut for squaring a binomial, which can be reversed to help us factor a perfect square trinomial.

Perfect Square Trinomial:

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

If we do not recognize a perfect square trinomial at first glance, we use the *ac method*. If we get two of the same numbers, we know we have a perfect square trinomial. Then we can factor using the square roots of the first and last terms, and the sign from the middle term.

Example 4. Factor completely.

$x^2 - 6x + 9$	Multiply to 9, sum to -6 Numbers are -3 and -3, the same; a perfect square trinomial Use square roots from first and last terms and sign from middle term
$= (x-3)^2$	Our Answer

Example 5. Factor completely.

$4x^2 + 20xy + 25y^2$	Multiply to 100, sum to 20 Numbers are 10 and 10, the same; perfect square trinomial Use square roots from first and last terms and sign from middle term
$= (2x+5y)^2$	Our Answer

SUM OR DIFFERENCE OF TWO PERFECT CUBES

Another special case involves the sum or difference of two perfect cubes. The sum and the difference of two perfect cubes have very similar factoring formulas:

Sum of Two Perfect Cubes:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Difference of Two Perfect Cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Start by expressing each term as the cube of a monomial. Use these results to determine the factored form of the expression. Comparing the formulas, you may notice that the only difference is the **signs** between the terms. One way to keep these two formulas straight is to think of **SOAP**.

> **S** stands for **S**ame sign as the original polynomial. If we have a sum of two perfect cubes, we add first; if we have a difference of two perfect cubes we subtract first.
> **O** stands for **O**pposite sign. If we have a sum, then subtraction is the second sign; a difference has addition for the second sign.
> **AP** stands for **A**lways **P**ositive. The last term for both formulas has an addition sign.

The following examples demonstrate factoring the sum or difference of two perfect cubes.

Example 6. Factor completely.

$m^3 - 27$	Express each term as the cube of a monomial
$= (m)^3 - (3)^3$	Apply the difference of two perfect cubes formula $(m \quad 3)(m^2 \quad 3m \quad 9)$; Use SOAP to fill in signs
$= (m-3)\,(m^2+3m+9)$	Our Answer

Example 7. Factor completely.

$125p^3 + 8r^3$	Express each term as the cube of a monomial
$= (5p)^3 + (2r)^3$	Apply the sum of two perfect cubes formula $(5p \quad 2r)\,(25p^2 \quad 10r \quad 4r^2)$; Use SOAP to fill in signs
$= (5p+2r)\,(25p^2 - 10pr + 4r^2)$	Our Answer

The previous example illustrates an important point. When we fill in the trinomial's first and last terms, we square the monomials $5p$ and $2r$. So, our squared terms in the second set of parentheses are $5p \cdot 5p = 25p^2$ and $2r \cdot 2r = 4r^2$. Notice that when done correctly, both the number and the variable are squared. Sometimes students forget to square both the number and the variable.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial, further. As a general rule, this factor will always be *prime* (unless there is a GCF that should have been factored before applying the appropriate perfect cubes rule).

SUMMARY OF FACTORING SPECIAL PRODUCTS

The following table summarizes all of the methods that we can use to factor special products:

FACTORING SPECIAL PRODUCTS

Difference of Squares:	$a^2 - b^2 = (a+b)(a-b)$
Sum of Squares:	prime
Perfect Square Trinomial:	$a^2 + 2ab + b^2 = (a+b)^2$
	$a^2 - 2ab + b^2 = (a-b)^2$
Sum of Cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes:	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

FACTORING USING MORE THAN ONE STRATEGY

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This process is shown in the following examples.

Example 8. Factor completely.

$72x^2 - 2$	GCF is 2; factor from each term
$= 2\,(36x^2 - 1)$	Difference of two perfect squares: $36x^2 = (6x)^2$ and $1 = (1)^2$
$= 2\,(6x+1)\,(6x-1)$	Our Answer

Example 9. Factor completely.

$48x^2y - 24xy + 3y$	GCF is $3y$; factor from each term
$= 3y\,(16x^2 - 8x + 1)$	Multiply to 16, sum to -8 Numbers are -4 and -4, the same; perfect square trinomial Use square roots from first and last terms and sign from middle term
$= 3y\,(4x-1)^2$	Our Answer

Example 10. Factor completely.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$; factor from each term
$= 2ab^2(64a^3 + 27b^3)$	Sum of two perfect cubes: $64a^3 = (4a)^3$ and $27b^3 = (3b)^3$
$= 2ab^2(4a+3b)\,(16a^2 - 12ab + 9b^2)$	Our Answer

Practice Exercises
Section 1.5: Factoring Special Products

Factor completely.

1) $x^2 - 49$
2) $x^2 - 9$
3) $v^2 - 25$
4) $1 - x^2$
5) $p^2 - 4$
6) $4v^2 - 1$
7) $64x^2 - 9y^2$
8) $9a^2 - 1$
9) $9x^2 + 1$
10) $3x^2 - 27$
11) $5n^2 - 20$
12) $16x^2 - 36$
13) $125x^2 + 45y^2$
14) $98a^2 - 50b^2$
15) $4m^2 + 64n^2$
16) $a^2 - 2a + 1$
17) $k^2 + 4k + 4$
18) $x^2 + 6x + 9$
19) $n^2 - 8n + 16$
20) $x^2 - 6x + 9$
21) $k^2 - 4k + 4$
22) $25p^2 - 10p + 1$
23) $x^2 + 2x + 1$
24) $25a^2 + 30ab + 9b^2$
25) $x^2 + 8xy + 16y^2$
26) $4a^2 - 20ab + 25b^2$
27) $49x^2 + 36y^2$
28) $8x^2 - 24xy + 18y^2$
29) $20x^2 + 20xy + 5y^2$
30) $x^3 - 8$
31) $x^3 + 64$
32) $x^3 - 64$
33) $x^3 + 8$
34) $216 - u^3$
35) $125x^3 - 216$
36) $125a^3 - 64$
37) $64x^3 - 27$
38) $64x^3 + 27y^3$
39) $32m^3 - 108n^3$
40) $54x^3 + 250y^3$

ANSWERS to Practice Exercises
Section 1.5: Factoring Special Products

1) $(x+7)(x-7)$
2) $(x+3)(x-3)$
3) $(v+5)(v-5)$
4) $(1+x)(1-x)$
5) $(p+2)(p-2)$
6) $(2v+1)(2v-1)$
7) $(8x+3y)(8x-3y)$
8) $(3a+1)(3a-1)$
9) prime
10) $3(x+3)(x-3)$
11) $5(n+2)(n-2)$
12) $4(2x+3)(2x-3)$
13) $5(25x^2+9y^2)$
14) $2(7a+5b)(7a-5b)$
15) $4(m^2+16n^2)$
16) $(a-1)^2$
17) $(k+2)^2$
18) $(x+3)^2$
19) $(n-4)^2$
20) $(x-3)^2$
21) $(k-2)^2$
22) $(5p-1)^2$
23) $(x+1)^2$
24) $(5a+3b)^2$
25) $(x+4y)^2$
26) $(2a-5b)^2$
27) prime
28) $2(2x-3y)^2$
29) $5(2x+y)^2$
30) $(x-2)(x^2+2x+4)$
31) $(x+4)(x^2-4x+16)$
32) $(x-4)(x^2+4x+16)$
33) $(x+2)(x^2-2x+4)$
34) $(6-u)(36+6u+u^2)$
35) $(5x-6)(25x^2+30x+36)$
36) $(5a-4)(25a^2+20a+16)$
37) $(4x-3)(16x^2+12x+9)$
38) $(4x+3y)(16x^2-12xy+9y^2)$
39) $4(2m-3n)(4m^2+6mn+9n^2)$
40) $2(3x+5y)(9x^2-15xy+25y^2)$

Section 1.6: Factoring Strategy

Objective: Identify and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which strategy to use and when. Here, we will organize all the different factoring methods we have seen.

A large part of deciding how to factor a polynomial is based the number of terms in the polynomial.

For all problem types, we will always try to factor out the GCF first.

FACTORING STRATEGY

1. If there is a common factor other than 1, factor the **Greatest Common Factor (GCF)**. Always look for the GCF first!

2. Count the number of terms. Select a method to try based on the number of terms:

   - **2 terms:** Try one of the special methods:
     Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
     Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
     Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

   - **3 terms:** Try the ***ac method***. Find two numbers that multiply to $a \cdot c$ and sum to b. Split the middle term and then continue to factor by grouping.
     If the trinomial is recognized as a perfect square trinomial, factor using one of these forms:
     $$a^2 + 2ab + b^2 = (a+b)^2 \quad \textbf{or} \quad a^2 - 2ab + b^2 = (a-b)^2$$

   - **4 terms:** Try factoring by **grouping**.

3. Check if the polynomial has been factored **completely**.

We will use the above strategy to factor each of the following examples. Here, the emphasis will be on which strategy to use rather than the steps used in that method.

Example 1. Factor completely.

$x^2 - 23x + 42$	GCF = 1, so no need to divide by GCF Three terms: try *ac method* Find factors to multiply to 42 and add to -23: Use -2 and -21
$= (x-2)(x-21)$	Our Answer

Example 2. Factor completely.

$z^2 + 6z - 9$	GCF = 1, so no need to divide by GCF Three terms: try *ac method* Find factors to multiply to -9 and add to 6 Try $(-1)\,(9)$, $(1)\,(-9)$, $(3)\,(-3)$; none sum to 6
prime	Our Answer

Example 3. Factor completely.

$4x^2 + 56xy + 196y^2$	GCF = 4, first factor GCF from each term
$= 4\,(x^2 + 14xy + 49y^2)$	Three terms inside parentheses: try *ac method* Find factors to multiply to 49 and add to 14 7 and 7; a perfect square trinomial
$= 4\,(x+7y)^2$	Our Answer

Example 4. Factor completely.

$5x^2y + 15xy - 35x^2 - 105x$	GCF = $5x$, first factor GCF from each term
$= 5x\,(xy + 3y - 7x - 21)$	Four terms inside parentheses: try grouping
$= 5x\,[y(x+3) - 7(x+3)]$	
$= 5x\,(x+3)\,(y-7)$	Our Answer

Example 5. Factor completely.

$100x^2 - 400$	GCF = 100, first factor GCF from both terms
$= 100\,(x^2 - 4)$	Two terms inside parentheses: try difference of two perfect squares
$= 100\,(x+2)\,(x-2)$	Our Answer

Example 6. Factor completely.

$108x^3y^2 - 39x^2y^2 + 3xy^2$	GCF = $3xy^2$, first factor GCF from each term
$= 3xy^2(36x^2 - 13x + 1)$	Three terms inside parentheses: try *ac method* Find factors to multiply to 36 and add to -13” Use -9 and -4

$= 3xy^2(36x^2 - 9x - 4x + 1)$	Factor by grouping
$= 3xy^2[9x(4x-1) - 1(4x-1)]$	
$= 3xy^2(4x-1)(9x-1)$	Our Answer

Example 7. Factor completely.

$5 + 625y^3$	GCF first, factor out 5 from each term
$= 5(1 + 125y^3)$	Two terms inside parentheses: try sum of two perfect cubes
$= 5(1+5y)(1-5y+25y^2)$	Our Answer

It is important to be comfortable and confident not just with using all the factoring methods, but also with deciding on which method to use. Your practice with these problems is very important!

Practice Exercises
Section 1.6: Factoring Strategy

Factor completely.

1) $24az-18ah+60yz-45hy$
2) $2x^2-11x+15$
3) $5u^2-9uv+4v^2$
4) $16x^2+48xy+36y^2$
5) $-2x^3+128y^3$
6) $20uv-60u^3-5xv+15xu^2$
7) $5n^3+7n^2-6n$
8) $2x^3+5x^2y+3y^2x$
9) $54u^3-16$
10) $54-128x^3$
11) n^2-n
12) $5x^2-22x-15$
13) $x^2-4xy+3y^2$
14) $45u^2-150uv+125v^2$
15) $64x^2+49y^2$
16) x^3-27y^3
17) m^2-4n^2
18) $12ab-18a+6bn-9n$
19) $36b^2c-16dx-24b^2d+24cx$
20) $3m^3-6m^2n-24mn^2$
21) $128+54x^3$
22) $64m^3+27n^3$
23) $2x^3+6x^2y-20xy^2$
24) $3ac+15ad^2+cx^2+5d^2x^2$
25) n^3+7n^2+10n
26) $64m^3-n^3$
27) $27x^3-64$
28) $16a^2-9b^2$
29) $5x^2+2x$
30) $2x^2-10x+12$
31) $-3k^3+27k^2-60k$
32) $75x^2-12y^2$
33) $mn-12x+3m-4nx$
34) $2k^2+k-10$
35) $16x^2-8xy+y^2$
36) v^2+v
37) $27m^2-48n^2$
38) x^3+4x^2
39) $9x^3+21x^2y-60xy^2$
40) $9n^3-3n^2$
41) $2m^2+6mn-20n^2$
42) $2u^2v^2-11uv^3+15v^4$
43) $5x^2-6x+7$
44) $9x^2-25y^2$
45) $2x^2-2x+14$
46) x^2-100

ANSWERS to Practice Exercises
Section 1.6: Factoring Strategy

1) $3(2a+5y)(4z-3h)$
2) $(2x-5)(x-3)$
3) $(5u-4v)(u-v)$
4) $4(2x+3y)^2$
5) $-2(x-4y)(x^2+4xy+16y^2)$
6) $5(4u-x)(v-3u^2)$
7) $n(5n-3)(n+2)$
8) $x(2x+3y)(x+y)$
9) $2(3u-2)(9u^2+6u+4)$
10) $2(3-4x)(9+12x+16x^2)$
11) $n(n-1)$
12) $(5x+3)(x-5)$
13) $(x-3y)(x-y)$
14) $5(3u-5v)^2$
15) prime
16) $(x-3y)(x^2+3xy+9y^2)$
17) $(m+2n)(m-2n)$
18) $3(2a+n)(2b-3)$
19) $4(3b^2+2x)(3c-2d)$
20) $3m(m+2n)(m-4n)$
21) $2(4+3x)(16-12x+9x^2)$
22) $(4m+3n)(16m^2-12mn+9n^2)$
23) $2x(x+5y)(x-2y)$
24) $(3a+x^2)(c+5d^2)$
25) $n(n+2)(n+5)$
26) $(4m-n)(16m^2+4mn+n^2)$
27) $(3x-4)(9x^2+12x+16)$
28) $(4a+3b)(4a-3b)$
29) $x(5x+2)$
30) $2(x-2)(x-3)$
31) $-3k(k-5)(k-4)$
32) $3(5x+2y)(5x-2y)$
33) $(m-4x)(n+3)$
34) $(2k+5)(k-2)$
35) $(4x-y)^2$
36) $v(v+1)$
37) $3(3m+4n)(3m-4n)$
38) $x^2(x+4)$
39) $3x(3x-5y)(x+4y)$
40) $3n^2(3n-1)$
41) $2(m-2n)(m+5n)$
42) $v^2(2u-5v)(u-3v)$
43) prime
44) $(3x+5y)(3x-5y)$
45) $2(x^2-x+7)$
46) $(x+10)(x-10)$

Section 1.7: Solving Equations by Factoring

Objective: Solve equations by factoring and using the zero product rule.

When solving linear equations such as $2x-5=21$, we can solve for the variable directly by adding 5 and dividing by 2 to get 13. However, when we have x^2 (or a higher power of x), we cannot just isolate the variable as we did with the linear equations. One method that we can use to solve for the variable will require us to use a property known as the zero product rule.

Zero Product Rule:

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

The zero product rule tells us that if two factors are multiplied together and the result is zero, then one of the factors must be zero.

USING THE ZERO PRODUCT RULE TO SOLVE EQUATIONS

We can use the zero product rule to help us solve equations having zero on one side and a factored expression on the other side as in the following example.

Example 1. Solve the equation.

$$(2x-3)(5x+1)=0 \quad \text{Set each factor equal to zero}$$

$$2x-3=0 \quad \text{or} \quad 5x+1=0 \quad \text{Solve each equation}$$

$$\underline{\quad +3\ +3} \qquad \underline{\quad -1\ -1}$$

$$\frac{2x}{2}=\frac{3}{2} \quad \text{or} \quad \frac{5x}{5}=\frac{-1}{5}$$

$$x=\frac{3}{2} \quad \text{or} \quad -\frac{1}{5} \quad \text{Our Solutions}$$

For the zero product rule to work, we must have factors to set equal to zero. This means if the expression is not already factored, we will need to factor it first, if possible.

Example 2. Solve the equation.

$$4x^2+x-3=0$$ Factor using the *ac* method: Find factors to multiply to -12 and add to 1

Use -3 and 4

$$4x^2-3x+4x-3=0$$ Factor by grouping

$$x(4x-3)+1(4x-3)=0$$

$$(4x-3)(x+1)=0$$ Set each factor equal to zero

$$4x-3=0 \quad \text{or} \quad x+1=0$$ Solve each equation

$$\underline{\;\;+3\;+3} \qquad \underline{\;-1\;\;-1}$$

$$\frac{4x}{4}=\frac{3}{4} \quad \text{or} \quad x=-1$$

$$x=\frac{3}{4} \quad \text{or} \quad -1$$ Our Solutions

Another important part of the zero product rule is that before we factor, one side of the equation must be zero. If one side of the equation is not zero, we must move terms around so that one side of the equation is zero. Generally, we like the coefficient of the x^2 term to be positive.

Example 3. Solve the equation.

$$x^2=8x-15$$ Set one side equal to 0, adding $-8x$ and 15 to both sides of the equation

$$\underline{-8x+15\quad -8x+15}$$

$$x^2-8x+15=0$$ Factor using the *ac method*: Find factors to multiply to 15 and add to -8

Use -5 and -3

$$(x-5)(x-3)=0$$ Set each factor equal to zero

$$x-5=0 \quad \text{or} \quad x-3=0$$ Solve each equation

$$\underline{\;+5\;\;+5} \qquad \underline{\;+3\;\;+3}$$

$$x=5 \quad \text{or} \quad x=3$$

$$x=5 \quad \text{or} \quad x=3$$ Our Solutions

Example 4. Solve the equation.

$$(x-7)(x+3)=-9$$ Not equal to zero; multiply first using FOIL

$$x^2-7x+3x-21=-9$$ Combine like terms

$$x^2-4x-21=-9$$ Set one side equal to 0 by adding 9 to both sides of the equation

$$\underline{+9=+9}$$

$$x^2-4x-12=0$$ Factor using the *ac method*: Find factors to multiply to -12 and add to -4

Use 6 and -2

$$(x-6)(x+2)=0$$ Set each factor equal to zero

$$x-6=0 \quad \text{or} \quad x+2=0$$ Solve each equation

$$\underline{\;+6\;\;+6} \qquad \underline{\;-2\;\;-2}$$

$$x=6 \quad \text{or} \quad x=-2$$ Our Solutions

Example 5. Solve the equation.

$$3x^2+4x-5=7x^2+4x-14$$ Set one side of the equation equal to 0

$$\underline{-3x^2-4x+5 \quad -3x^2-4x+5}$$

$$0=4x^2-9$$ Factor using the difference of two perfect squares

$$0=(2x+3)(2x-3)$$ Set each factor equal to zero

$$2x+3=0 \quad \text{or} \quad 2x-3=0$$ Solve each equation

$$\underline{-3 \quad -3} \qquad \underline{+3 \quad +3}$$

$$\frac{2x}{2}=\frac{-3}{2} \quad \text{or} \quad \frac{2x}{2}=\frac{3}{2}$$

$$x=-\frac{3}{2} \quad \text{or} \quad x=\frac{3}{2}$$ Our Solutions

Most quadratic equations (equations where the highest exponent of the variable is 2) have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 6. Solve the equation.

$$4x^2=12x-9$$ Set one side of the equation equal to 0

$$\underline{-12x+9 \quad -12x+9}$$

$$4x^2-12x+9=0$$ Factor using the *ac method*: Find factors to multiply to 36 and add to -12

Use -6 and -6 (the same); a perfect square trinomial

$$(2x-3)^2=0$$ Since $(2x-3)^2=(2x-3)(2x-3)$, both factors are identical. Set the factor $(2x-3)$ equal to zero.

$$2x-3=0$$ Solve the equation

$$\underline{+3 \quad +3}$$

$$\frac{2x}{2}=\frac{3}{2}$$

$$x=\frac{3}{2}$$ Our Solution

As always, it will be important to factor out the GCF first if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This might give us more than just two solutions. The next few examples illustrate this.

Example 7. Solve the equation.

$$4x^2=8x$$ Set equal to 0 by subtracting $8x$ from both sides

$$\underline{-8x \quad -8x}$$

$$4x^2-8x=0$$ Factor the GCF of $4x$

$$4x(x-2)=0$$ Set each factor equal to zero

$$\frac{4x}{4}=\frac{0}{4} \quad \text{or} \quad x-2=0$$ Solve each equation

$$x=0 \quad \text{or} \quad \begin{array}{r} +2 \quad +2 \\ \hline x=2 \end{array}$$

$$x=0 \quad \text{or} \quad x=2$$ Our Solutions

Example 8. Solve the equation.

$$2x^3-14x^2+24x=0$$ Factor the GCF of $2x$

$$2x(x^2-7x+12)=0$$ Factor with *ac method*: Find factors to multiply to 12 and add to -7

Use -3 and -4

$$2x(x-3)(x-4)=0$$ Set each factor equal to zero

$$\frac{2x}{2}=\frac{0}{2} \quad \text{or} \quad x-3=0 \quad \text{or} \quad x-4=0$$ Solve each equation

$$x=0 \qquad \begin{array}{r} +3 \quad +3 \\ \hline x=3 \end{array} \qquad \begin{array}{r} +4 \quad +4 \\ \hline x=4 \end{array}$$

$$x=0,\ 3, \text{or } 4$$ Our Solutions

Example 9. Solve the equation.

$$6x^2+21x-27=0$$ Factor the GCF of 3

$$3(2x^2+7x-9)=0$$ Factor with *ac method*: Find factors to multiply to -18 and add to 7

Use 9 and -2

$$3(2x^2+9x-2x-9)=0$$ Factor by grouping

$$3[x(2x+9)-1(2x+9)]=0$$

$$3(2x+9)(x-1)=0$$ Set each factor equal to zero

$$3=0 \quad \text{or} \quad 2x+9=0 \quad \text{or} \quad x-1=0$$ Solve each equation

$$\begin{array}{r} -9=-9 \\ \hline \end{array} \qquad \begin{array}{r} +1 \quad +1 \\ \hline \end{array}$$

$$3\neq 0 \quad \text{or} \quad \frac{2x}{2}=\frac{-9}{2} \quad \text{or} \quad x=1$$

$$x=-\frac{9}{2}$$

$$x=-\frac{9}{2} \quad \text{or} \quad x=1$$ Our Solutions

In the previous example, the GCF did not contain a variable. When we set this factor equal to zero, we get a false statement. No solution comes from this factor. When the GCF has no variables, we may skip setting the GCF equal to zero.

Just as not all polynomials can be factored, not all equations can be solved by factoring. If an equation cannot be solved by factoring, we will have to use another method. These other methods are saved for another lesson.

APPLICATIONS OF SOLVING EQUATIONS

In science, we often use a mathematical model to describe a physical situation. To answer questions about the situation, we may need to set up and solve an equation. In this section, we will be able to solve the equations by factoring.

Example 10. Bob is on the balcony of his apartment, which is 80 feet above the ground. He tosses a ball vertically upward. The ball's height above the ground as it travels is modeled by the equation $h = -16t^2 + 64t + 80$ where t is the time (in seconds) the ball has been in flight and h is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

$h = -16t^2 + 64t + 80$	The time that has passed, t, is unknown; when the ball hits the ground, its height h is zero
$0 = -16t^2 + 64t + 80$	Set the quadratic equation equal to zero and solve by factoring
$0 = -16(t^2 - 4t - 5)$	Factor the GCF of -16
$0 = -16(t-5)(t+1)$	Factor with *ac method:* Find factors to multiply to -5 and add to -4 Use -5 and 1
$-16 = 0$ or $t - 5 = 0$ or $t + 1 = 0$	Set each factor equal to zero
$+5 \; +5$ $\qquad$ $-1 \; -1$	
$-16 \neq 0$ or $t = 5$ or $t = -1$	
$t = 5$ or $t = -1$	Time cannot be negative; so $t = -1$ extraneous (not considered to be a solution to the equation in this context.
The ball hits the ground after 5 seconds.	Our Solution

Practice Exercises
Section 1.7: Solving Equations by Factoring

Solve each equation by factoring.

1) $(x-1)(x+4)=0$
2) $0=(2x+5)(x-7)$
3) $x^2-4=0$
4) $2x^2-18x=0$
5) $6x^2-150=0$
6) $p^2+4p-32=0$
7) $2n^2+10n-28=0$
8) $m^2-m-30=0$
9) $7x^3+26x^2+15x=0$
10) $-16t^2+24t+16=0$
11) $x^2-4x-8=-8$
12) $x^2-5x-1=-5$
13) $a^3-6a^2+6a=-2a$
14) $7x^2+17x-20=-8$
15) $4n^2-13n+8=5$
16) $7r^2+84=-49r$
17) $x^2-6x=-9$
18) $7n^2-28n=0$
19) $3v^2+7v=40$
20) $6b^2=5+7b$
21) $35x^2+120x=-45$
22) $3n^2+3n=6$
23) $k^2+24k+89=6k+8$
24) $a^2+7a-9=-3+6a$
25) $9x^2-46+7x=7x+8x^2+3$
26) $x^2+10x+30=6$
27) $2m^2+19m+40=-2m$
28) $5n^2+41n+40=-2$
29) $24x^2+11x-80=3x$
30) $121w^2+8w-7=8w-6$

Solve each application.

31) A ball is tossed vertically upward from a building which is 96 feet above the ground. The ball's height above the ground as it travels is modeled by the equation $h=-16t^2+16t+96$ where t is the time (in seconds) the ball has been in flight and h is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

32) An explosion causes debris to fly vertically upward with an initial speed of 80 feet per second. The height of the debris above the ground is modeled by the equation $h=-16t^2+80t$ where t is the time (in seconds) after the explosion and h is the height of the debris (in feet) at any particular time. How long does it take for the debris to hit the ground?

ANSWERS to Practice Exercises
Section 1.7: Solving Equations by Factoring

1) $1, -4$
2) $-\frac{5}{2}, 7$
3) $-2, 2$
4) $0, 9$
5) $-5, 5$
6) $4, -8$
7) $2, -7$
8) $-5, 6$
9) $-\frac{5}{7}, -3, 0$
10) $-\frac{1}{2}, 2$
11) $4, 0$
12) $1, 4$
13) $0, 4, 2$
14) $\frac{4}{7}, -3$
15) $\frac{1}{4}, 3$
16) $-4, -3$
17) 3
18) $4, 0$
19) $\frac{8}{3}, -5$
20) $-\frac{1}{2}, \frac{5}{3}$
21) $-\frac{3}{7}, -3$
22) $-2, 1$
23) -9
24) $2, -3$
25) $-7, 7$
26) $-4, -6$
27) $-\frac{5}{2}, -8$
28) $-\frac{6}{5}, -7$
29) $\frac{5}{3}, -2$
30) $-\frac{1}{11}, \frac{1}{11}$

31) The ball hits the ground after 3 seconds.

32) The debris hits the ground after 5 seconds.

Review: Chapter 1

Factor completely.

1) $20x^3 + 15x^2 + 35x$
2) $w^3 + 3w^2 - 2w - 6$
3) $z^2 + 8z - 20$
4) $40z^{18} + 5z^{12} + 10z^6$
5) $125x^3 + 64y^3$
6) $80x^2 - 45y^2$
7) $25x^2 + 9$
8) $10k^2 - k - 3$
9) $4y^2 - 23y - 35$
10) $4x^2 - 12x + 9$
11) $8x^3 - 1$
12) $-2x^3 + 8x^2 - 6x$
13) $45w^2 + 6w - 3$

Solve.

14) $(2x-3)(x+2) = 0$
15) $m^2 - 3m = 0$
16) $p^2 = -6p + 27$
17) $x^2 - 3x + 1 = 5$
18) $a^3 - a^2 - 6a = -4a$
19) $25x^2 = 16$
20) A ball is tossed vertically upward from a building which is 96 feet above the ground. The ball's height above the ground as it travels is modeled by the equation $h = -16t^2 + 64t + 80$ where t is the time (in seconds) the ball has been in flight and h is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

ANSWERS to Review: Chapter 1

1) $5x(4x^2+3x+7)$
2) $(w+3)(w^2-2)$
3) $(z+10)(z-2)$
4) $5z^6(8z^{12}+z^6+2)$
5) $(5x+4y)(25x^2-20xy+16y^2)$
6) $5(4x+3y)(4x-3y)$
7) prime
8) $(5k-3)(2k+1)$
9) $(4y+5)(y-7)$
10) $(2x-3)^2$
11) $(2x-1)(4x^2+2x+1)$
12) $-2x(x-1)(x-3)$
13) $3(5w-1)(3w+1)$
14) $-2, \frac{3}{2}$
15) $0, 3$
16) $-9, 3$
17) $-1, 4$
18) $-1, 0, 2$
19) $-\frac{4}{5}, \frac{4}{5}$
20) The ball hits the ground after 5 seconds.