CHAPTER 3

Radical Expressions and Equations

Objectives
Chapter 3

- Simplify expressions with square roots.
- Simplify radicals with an index greater than two.
- Add and subtract like radicals by first simplifying each radical.
- Multiply and divide radical expressions using the product and quotient rules for radicals.
- Rationalize the denominators of radical expressions.
- Convert between radical notation and exponential notation.
- Simplify expressions with rational exponents using the properties of exponents.
- Multiply and divide radical expressions with different indices.
- Solve equations with radicals and check for extraneous solutions.
- Add, subtract, multiply, divide, and simplify expressions using complex numbers.

Section 3.1: Square Roots

Objective: Simplify expressions with square roots.

To reverse the process of squaring a number, we find the square root of a number. In other words, a square root "un-squares" a number.

PRINCIPAL SQUARE ROOT

If a is a n*onnegative* real number, then the **principal square root** of a is the *nonnegative* number b such that $b^2 = a$.
We write $b = \sqrt{a}$.

The symbol $\sqrt{\ }$ is called the *radical sign* and the number a, under the radical sign, is called the *radicand*. An expression containing a radical sign is called a *radical expression*. Square roots are the most common type of radical expressions used.

The following example shows several square roots:

Example 1. Evaluate.

$\sqrt{1} = 1$ because $1^2 = 1$	$\sqrt{121} = 11$ because $11^2 = 121$
$\sqrt{4} = 2$ because $2^2 = 4$	$\sqrt{625} = 25$ because $25^2 = 625$
$\sqrt{9} = 3$ because $3^2 = 9$	$\sqrt{0} = 0$ because $0^2 = 0$
$\sqrt{16} = 4$ because $4^2 = 16$	$\sqrt{81} = 9$ because $9^2 = 81$
$\sqrt{25} = 5$ because $5^2 = 25$	$\sqrt{-81}$ is not a real number

Notice that $\sqrt{-81}$ is not a real number because there is no real number whose square is -81. If we square a positive number or a negative number, the result will always be positive. Thus, we can only take square roots of nonnegative numbers. In another section, we will define a method we can use to work with and evaluate square roots of negative numbers, but for now we will state they are not real numbers.

We call numbers like 1, 4, 9, 16, 25, 81, 121, and 625 *perfect squares* because they are squares of integers. Not all numbers are perfect squares. For example, 8 is not a perfect square because 8 is not the square of an integer. Using a calculator, $\sqrt{8}$ is approximately equal to 2.828427125... and that number is still a rounded approximation of the square root.

SIMPLIFYING SQUARE ROOTS

Instead of using decimal approximations, we will usually express roots in *simplest* radical form. Advantages of simplest radical form are that it is an exact answer (not an approximation) and that calculations and algebraic manipulations can be done more easily.

To express roots in simplest radical form, we will use the following property:

PRODUCT RULE OF SQUARE ROOTS

For any *nonnegative* real numbers a and b,

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$$

When simplifying a square root expression, we will first find the largest perfect square factor of the radicand. Then, we will write the radicand as the product of two factors, apply the product rule, and evaluate the square root of the perfect square factor.

Example 2. Simplify.

$\sqrt{75}$	75 is divisible by the perfect square 25; Split radicand into factors
$=\sqrt{25\cdot 3}$	Apply product rule
$=\sqrt{25}\cdot\sqrt{3}$	Take the square root of 25
$=5\sqrt{3}$	Our Answer

If there is a coefficient in front of the radical to begin with, the problem becomes a big multiplication problem.

Example 3. Simplify.

$5\sqrt{63}$	63 is divisible by the perfect square 9; Split radicand into factors
$=5\sqrt{9\cdot 7}$	Apply product rule
$=5\sqrt{9}\cdot\sqrt{7}$	Take the square root of 9
$=5\cdot 3\sqrt{7}$	Multiply coefficients
$=15\sqrt{7}$	Our Answer

As we simplify radicals using this method, it is important to be sure our final answer can be simplified no more.

Example 4. Simplify.

$\sqrt{72}$	72 is divisible by the perfect square 9; Split radicand into factors
$=\sqrt{9\cdot 8}$	Apply product rule
$=\sqrt{9}\cdot\sqrt{8}$	Take the square root of 9
$=3\sqrt{8}$	But 8 is also divisible by the perfect square 4; Split radicand into factors
$=3\sqrt{4\cdot 2}$	Apply product rule
$=3\sqrt{4}\cdot\sqrt{2}$	Take the square root of 4
$=3\cdot 2\sqrt{2}$	Multiply
$=6\sqrt{2}$	Our Answer

The previous example could have been done in fewer steps if we had noticed that $72=36\cdot 2$, where 36 is the largest perfect square factor of 72. Often the time it takes to discover the larger perfect square is more than it would take to simplify the radicand in several steps.

Variables are often part of the radicand as well. To simplify radical expressions involving variables, use the property below:

> **SIMPLIFYING $\sqrt{a^2}$**
>
> For any *nonnegative* real number a,
>
> $$\sqrt{a^2}=a$$

Note this property only holds if *a* is *nonnegative*. For this reason, we will assume that all variables involved in a radical expression are *nonnegative*.

When simplifying with variables, variables with exponents that are divisible by 2 are perfect squares. For example, by the power of a power rule of exponents, $(x^4)^2=x^8$. So x^8 is a perfect square and $\sqrt{x^8}=\sqrt{(x^4)^2}=x^4$. A shortcut for taking the square roots of variables is to divide the exponent by 2. In our example, $\sqrt{x^8} = x^4$ because we divide the exponent 8 by 2 to get 4. When squaring, we multiply the exponent by 2, so when taking a square root, we divide the exponent by 2.

This process is shown in the following example.

Example 5. Simplify.

$-5\sqrt{18x^4y^6z^{10}}$	18 is divisible by the perfect square 9; Split radicand into factors
$=-5\sqrt{9\cdot 2x^4y^6z^{10}}$	Apply product rule
$=-5\sqrt{9}\cdot\sqrt{2}\cdot\sqrt{x^4}\cdot\sqrt{y^6}\cdot\sqrt{z^{10}}$	Simplify roots; divide exponents by 2
$=-5\cdot 3x^2y^3z^5\sqrt{2}$	Multiply coefficients
$=-15x^2y^3z^5\sqrt{2}$	Our Answer

We can't always evenly divide the exponent of a variable by 2. Sometimes we have a remainder. If there is a remainder, this means the variable with an exponent equal to the remainder will remain inside the radical sign. On the outside of the radical, the exponent of the variable will be equal to the whole number part. This process is shown in the following example.

Example 6. Simplify.

$\sqrt{20x^5y^9z^6}$	20 is divisible by the perfect square 4; Split radicand into factors
$=\sqrt{4\cdot 5x^5y^9z^6}$	Apply product rule
$=\sqrt{4}\cdot\sqrt{5}\cdot\sqrt{x^5}\cdot\sqrt{y^9}\cdot\sqrt{z^6}$	Simplify roots; divide exponents by 2, remainder is left inside
$=2x^2y^4z^3\sqrt{5xy}$	Our Answer

In the previous example, for the variable *x*, we divided $\frac{5}{2} = 2\,R\,1$, so x^2 came out of the radicand and $x^1 = x$ remained inside the radicand. For the variable y, we divided $\frac{9}{2} = 4\,R\,1$, so y^4 came out of the radicand and $y^1 = y$ remained inside. For the variable z, we divided $\frac{6}{2} = 3\,R\,0$, so z^3 came out of the radicand and no zs remained inside.

Practice Exercises
Section 3.1: Square Roots

Simplify. Assume that all variables represent nonnegative real numbers.

1) $\sqrt{36}$
2) $\sqrt{-100}$
3) $-\sqrt{196}$
4) $\sqrt{12}$
5) $\sqrt{125}$
6) $\sqrt{72}$
7) $\sqrt{245}$
8) $3\sqrt{24}$
9) $5\sqrt{48}$
10) $6\sqrt{128}$
11) $-8\sqrt{392}$
12) $-7\sqrt{63}$
13) $\sqrt{192n}$
14) $\sqrt{343b}$
15) $\sqrt{169v^2}$
16) $\sqrt{100n^3}$
17) $\sqrt{252x^2}$
18) $\sqrt{200a^3}$
19) $-\sqrt{100k^4}$
20) $-4\sqrt{175p^4}$
21) $-7\sqrt{64x^4}$
22) $-5\sqrt{36m}$
23) $\sqrt{45x^2y^2}$
24) $\sqrt{72a^3b^4}$
25) $\sqrt{16x^3y^3}$
26) $\sqrt{98a^4b^2}$
27) $\sqrt{320x^4y^4}$
28) $\sqrt{512m^4n^3}$
29) $6\sqrt{80xy^2}$
30) $8\sqrt{98mn}$
31) $5\sqrt{245x^2y^3}$
32) $2\sqrt{72x^2y^2}$
33) $-2\sqrt{180u^3v}$
34) $-5\sqrt{28x^3y^4}$
35) $-8\sqrt{108x^4y^2z^4}$
36) $6\sqrt{50a^4bc^2}$
37) $2\sqrt{80hj^4k}$
38) $-\sqrt{32xy^2z^3}$
39) $-4\sqrt{54mnp^2}$
40) $-8\sqrt{56m^2p^4q}$

ANSWERS to Practice Exercises
Section 3.1: Square Roots

1) 6
2) not a real number
3) -14
4) $2\sqrt{3}$
5) $5\sqrt{5}$
6) $6\sqrt{2}$
7) $7\sqrt{5}$
8) $6\sqrt{6}$
9) $20\sqrt{3}$
10) $48\sqrt{2}$
11) $-112\sqrt{2}$
12) $-21\sqrt{7}$
13) $8\sqrt{3n}$
14) $7\sqrt{7b}$
15) $13v$
16) $10n\sqrt{n}$
17) $6x\sqrt{7}$
18) $10a\sqrt{2a}$
19) $-10k^2$
20) $-20p^2\sqrt{7}$
21) $-56x^2$
22) $-30\sqrt{m}$
23) $3xy\sqrt{5}$
24) $6ab^2\sqrt{2a}$
25) $4xy\sqrt{xy}$
26) $7a^2b\sqrt{2}$
27) $8x^2y^2\sqrt{5}$
28) $16m^2n\sqrt{2n}$
29) $24y\sqrt{5x}$
30) $56\sqrt{2mn}$
31) $35xy\sqrt{5y}$
32) $12xy\sqrt{2}$
33) $-12u\sqrt{5uv}$
34) $-10xy^2\sqrt{7x}$
35) $-48x^2yz^2\sqrt{3}$
36) $30a^2c\sqrt{2b}$
37) $8j^2\sqrt{5hk}$
38) $-4yz\sqrt{2xz}$
39) $-12p\sqrt{6mn}$
40) $-16mp^2\sqrt{14q}$

Section 3.2: Higher Roots

Objective: Simplify radicals with an index greater than two.

While square roots are the most common type of radical we work with, we can take higher roots of numbers as well: cube roots, fourth roots, fifth roots, etc. Following is a definition of higher roots.

n^{th} ROOT

For a positive integer $n > 1$, the principal n^{th} root of a is the number b such that $b^n = a$.
We write $b = \sqrt[n]{a}$.

NOTE: If *n* is even, *a* and *b* are *nonnegative*.

We call b the n^{th} root of a. The small letter n is called the **index**. It tells us which root we are taking, or which power we are "un-doing". For square roots, the index is 2. As this is the most common root, the two is not usually written.

The following example includes several higher roots.

Example 1. Evaluate.

$\sqrt[3]{125} = 5$ because $5^3 = 125$	$\sqrt[3]{-64} = -4$ because $(-4)^3 = -64$
$\sqrt[3]{8} = 2$ because $2^3 = 8$	$\sqrt[7]{-128} = -2$ because $(-2)^7 = -128$
$\sqrt[4]{81} = 3$ because $3^4 = 81$	$\sqrt[4]{16} = 2$ because $2^4 = 16$
$\sqrt[5]{32} = 2$ because $2^5 = 32$	$\sqrt[4]{-16}$ is not a real number

We must be careful of a few things as we work with higher roots. First, it is important to check the index on the root. For example, $\sqrt{81} = 9$ because $9^2 = 81$ but $\sqrt[4]{81} = 3$ because $3^4 = 81$. Another thing to watch out for is negative numbers in the radicand. We can take an odd root of a negative number because a negative number raised to an odd power is still negative. However, the even root of a negative number is not a real number. In a later section we will discuss how to work with even roots of negative numbers, but for now we state they are not real numbers.

SIMPLIFYING HIGHER ROOTS

We can simplify higher roots in much the same way we simplified square roots, using the product rule of radicals.

PRODUCT RULE OF RADICALS

For any *nonnegative* real numbers a and b,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

Often, we are not as familiar with perfect nth powers as we are with perfect squares. It is important to remember what index we are working with as we express higher roots in simplest radical form.

Example 2. Simplify.

$\sqrt[3]{54}$	We are working with a cube root, want perfect third powers. Test 2: $2^3 = 8$; 54 is not divisible by 8. Test 3: $3^3 = 27$; 54 is divisible by 27! Split radicand into factors
$= \sqrt[3]{27 \cdot 2}$	Apply product rule
$= \sqrt[3]{27} \cdot \sqrt[3]{2}$	Take the cube root of 27
$= 3\sqrt[3]{2}$	Our Answer

Just as with square roots, if we have a coefficient, then we multiply the new coefficients together.

Example 3. Simplify.

$3\sqrt[4]{48}$	We are working with a fourth root, want perfect fourth powers. Test 2: $2^4 = 16$; 48 is divisible by 16! Split radicand into factors
$= 3\sqrt[4]{16 \cdot 3}$	Apply product rule
$= 3\sqrt[4]{16} \cdot \sqrt[4]{3}$	Take the fourth root of 16
$= 3 \cdot 2\sqrt[4]{3}$	Multiply coefficients
$= 6\sqrt[4]{3}$	Our Answer

We can also take higher roots of variables. To simplify radical expressions involving variables, use the property below:

SIMPLIFYING $\sqrt[n]{a^n}$

For any *nonnegative* real number a,

$$\sqrt[n]{a^n} = a$$

Note this property only holds if *a* is *nonnegative*. For this reason, we will assume that all variables involved in a radical expression are *nonnegative*.

As with square roots, when simplifying with variables, we will divide the variable's exponent by the index. The whole number part of the division is how many factors of that variable will come out of the nth root. Any remainder is how many factors of the variable are left behind in the radicand. This process is shown in the following examples.

Example 4. Simplify.

$\sqrt[5]{x^{25}y^{17}z^3}$	Divide each exponent by 5: whole number outside, remainder inside
$= x^5y^3\sqrt[5]{y^2z^3}$	Our Answer

In the previous example, for the variable x, we divided $\frac{25}{5} = 5R0$, so x^5 came out and no xs remained inside. For y, we divided $\frac{17}{5} = 3R2$, so y^3 came out, and y^2 remained inside. For z, when we divided $\frac{3}{5} = 0R3$, all three or z^3 remained inside.

Example 5. Simplify.

$2\sqrt[3]{40a^4b^8}$	40 is divisible by the perfect cube 8 Split radicand into factors Apply product rule
$= 2 \cdot \sqrt[3]{8} \cdot \sqrt[3]{5} \cdot \sqrt[3]{a^4} \cdot \sqrt[3]{b^8}$	Simplify roots; divide exponents by 3, remainders are left inside
$= 2 \cdot 2ab^2\sqrt[3]{5ab^2}$	Multiply coefficients
$= 4ab^2\sqrt[3]{5ab^2}$	Our Answer

Practice Exercises
Section 3.2: Higher Roots

Simplify.

1) $\sqrt[3]{64}$
2) $\sqrt[3]{-27}$
3) $\sqrt[4]{16}$
4) $\sqrt[4]{-16}$
5) $\sqrt[5]{-1}$
6) $\sqrt[8]{-1}$
7) $\sqrt[3]{625}$
8) $\sqrt[3]{375}$
9) $\sqrt[3]{750}$
10) $\sqrt[3]{250}$
11) $\sqrt[3]{24}$
12) $-4\sqrt[4]{96}$
13) $3\sqrt[4]{48}$
14) $-\sqrt[4]{112}$
15) $5\sqrt[4]{243}$
16) $\sqrt[4]{648a^2}$
17) $\sqrt[4]{64n^3}$
18) $\sqrt[5]{224n^3}$
19) $\sqrt[5]{-96x^3}$
20) $\sqrt[5]{224p^5}$
21) $\sqrt[6]{256x^6}$
22) $-8\sqrt[7]{384b^8}$
23) $-2\sqrt[3]{-48v^7}$
24) $-7\sqrt[3]{320n^6}$
25) $-\sqrt[3]{512n^6}$
26) $\sqrt[3]{-135x^5y^3}$
27) $\sqrt[3]{64u^5v^3}$
28) $\sqrt[3]{-32x^4y^4}$
29) $\sqrt[3]{1000a^4b^5}$
30) $\sqrt[3]{256x^4y^6}$
31) $7\sqrt[3]{-81x^3y^7}$
32) $-4\sqrt[3]{56x^2y^8}$
33) $8\sqrt[3]{-750xy}$
34) $-3\sqrt[3]{192ab^2}$
35) $3\sqrt[3]{135xy^3}$
36) $6\sqrt[3]{-54m^8n^3p^7}$
37) $-8\sqrt[4]{80m^4p^7q^4}$
38) $-2\sqrt[4]{405a^5b^8c}$
39) $7\sqrt[4]{128h^6j^8k^8}$
40) $5\sqrt[4]{324x^7yz^7}$

ANSWERS to Practice Exercises
Section 3.2: Higher Roots

1) 4
2) -3
3) 2
4) not a real number
5) -1
6) not a real number
7) $5\sqrt[3]{5}$
8) $5\sqrt[3]{3}$
9) $5\sqrt[3]{6}$
10) $5\sqrt[3]{2}$
11) $2\sqrt[3]{3}$
12) $-8\sqrt[4]{6}$
13) $6\sqrt[4]{3}$
14) $-2\sqrt[4]{7}$
15) $15\sqrt[4]{3}$
16) $3\sqrt[4]{8a^2}$
17) $2\sqrt[4]{4n^3}$
18) $2\sqrt[5]{7n^3}$
19) $-2\sqrt[5]{3x^3}$
20) $2p\sqrt[5]{7}$
21) $2x\sqrt[6]{4}$
22) $-16b\sqrt[7]{3b}$
23) $4v^2\sqrt[3]{6v}$
24) $-28n^2\sqrt[3]{5}$
25) $-8n^2$
26) $-3xy\sqrt[3]{5x^2}$
27) $4uv\sqrt[3]{u^2}$
28) $-2xy\sqrt[3]{4xy}$
29) $10ab\sqrt[3]{ab^2}$
30) $4xy^2\sqrt[3]{4x}$
31) $-21xy^2\sqrt[3]{3y}$
32) $-8y^2\sqrt[3]{7x^2y^2}$
33) $-40\sqrt[3]{6xy}$
34) $-12\sqrt[3]{3ab^2}$
35) $9y\sqrt[3]{5x}$
36) $-18m^2np^2\sqrt[3]{2m^2p}$
37) $-16mpq\sqrt[4]{5p^3}$
38) $-6ab^2\sqrt[4]{5ac}$
39) $14hj^2k^2\sqrt[4]{8h^2}$
40) $15xz\sqrt[4]{4x^3yz^3}$

Section 3.3: Add and Subtract Radical Expressions

Objective: Add and subtract like radicals by first simplifying each radical.

Adding and subtracting radical expressions is very similar to adding and subtracting variable expressions. If two or more radical expressions have the same indices and the same radicands, they are called *like radicals*. Consider the similarities between the following two examples.

Example 1. Perform the indicated operations.

$5x + 3x - 2x$ Combine like terms

$= 6x$ Our Answer

Example 2. Perform the indicated operations.

$5\sqrt{11} + 3\sqrt{11} - 2\sqrt{11}$ Combine like radicals

$= 6\sqrt{11}$ Our Answer

Notice that when we combined the radical terms with $\sqrt{11}$ it was just like combining variable terms with x. When adding and subtracting like radicals, we add and subtract the coefficients in front of the radical, and the radical stays the same.

Example 3. Perform the indicated operations.

$7\sqrt[5]{6} + 4\sqrt[5]{3} - 9\sqrt[5]{3} + \sqrt[5]{6}$ Combine like radicals $7\sqrt[5]{6} + \sqrt[5]{6}$ and $4\sqrt[5]{3} - 9\sqrt[5]{3}$

$= 8\sqrt[5]{6} - 5\sqrt[5]{3}$ Our Answer

We cannot combine these radical expressions any more because the radicals are not like radical terms.

Often radical expressions do not look *like* at first. However, if we simplify the radicals, we may find we do in fact have like radicals. This process is shown in the examples on the next page.

Example 4. Perform the indicated operations.

$5\sqrt{45}+6\sqrt{18}-2\sqrt{98}+\sqrt{20}$	Simplify radicals, finding perfect square factors
$=5\sqrt{9\cdot 5}+6\sqrt{9\cdot 2}-2\sqrt{49\cdot 2}+\sqrt{4\cdot 5}$	Take square roots where possible
$=5\cdot 3\sqrt{5}+6\cdot 3\sqrt{2}-2\cdot 7\sqrt{2}+2\sqrt{5}$	Multiply coefficients
$=15\sqrt{5}+18\sqrt{2}-14\sqrt{2}+2\sqrt{5}$	Combine like radicals
$=17\sqrt{5}+4\sqrt{2}$	Our Answer

Example 5. Perform the indicated operations.

$4\sqrt[3]{54}-9\sqrt[3]{16}+5\sqrt[3]{9}$	Simplify radicals, finding perfect cube factors
$=4\sqrt[3]{27\cdot 2}-9\sqrt[3]{8\cdot 2}+5\sqrt[3]{9}$	Take cube roots where possible
$=4\cdot 3\sqrt[3]{2}-9\cdot 2\sqrt[3]{2}+5\sqrt[3]{9}$	Multiply coefficients
$=12\sqrt[3]{2}-18\sqrt[3]{2}+5\sqrt[3]{9}$	Combine like radicals $12\sqrt[3]{2}-18\sqrt[3]{2}$
$=-6\sqrt[3]{2}+5\sqrt[3]{9}$	Our Answer

Practice Exercises
Section 3.3: Add and Subtract Radical Expressions

Perform the indicated operation.

1) $2\sqrt{5}+2\sqrt{5}+2\sqrt{5}$

2) $-3\sqrt{6}-3\sqrt{3}-2\sqrt{3}$

3) $-3\sqrt{2}+3\sqrt{5}+3\sqrt{5}$

4) $-2\sqrt{6}-\sqrt{3}+3\sqrt{6}$

5) $-2\sqrt{6}-2\sqrt{6}-\sqrt{6}$

6) $-3\sqrt{3}+5\sqrt{3}+2\sqrt{3}$

7) $3\sqrt{6}+3\sqrt{5}+2\sqrt{5}$

8) $-\sqrt{5}+2\sqrt{3}-2\sqrt{3}$

9) $2\sqrt{2}-3\sqrt{18}-\sqrt{2}$

10) $-\sqrt{54}-3\sqrt{6}+3\sqrt{27}$

11) $-3\sqrt{6}-\sqrt{12}+3\sqrt{3}$

12) $4\sqrt{5}-\sqrt{5}-2\sqrt{48}$

13) $3\sqrt{2}+2\sqrt{8}-3\sqrt{18}$

14) $2\sqrt{20}+2\sqrt{20}-\sqrt{3}$

15) $3\sqrt{18}-\sqrt{2}-3\sqrt{2}$

16) $3\sqrt{27}+2\sqrt{3}-\sqrt{12}$

17) $-2\sqrt{2}-\sqrt{2}+3\sqrt{8}+3\sqrt{6}$

18) $-2\sqrt{18}-3\sqrt{8}-\sqrt{20}+2\sqrt{20}$

19) $-3\sqrt{18}-\sqrt{8}+5\sqrt{8}+2\sqrt{8}$

20) $-2\sqrt{24}-2\sqrt{6}+2\sqrt{6}+2\sqrt{20}$

21) $-3\sqrt{8}-\sqrt{5}-3\sqrt{6}+2\sqrt{18}$

22) $3\sqrt{24}-3\sqrt{27}+2\sqrt{6}+2\sqrt{8}$

23) $2\sqrt{6}-\sqrt{54}-3\sqrt{27}-\sqrt{3}$

24) $-2\sqrt[3]{16}+2\sqrt[3]{16}+2\sqrt[3]{2}$

25) $3\sqrt[3]{135}-\sqrt[3]{81}-\sqrt[3]{135}$

26) $2\sqrt[4]{243}-2\sqrt[4]{243}-\sqrt[4]{3}$

27) $-3\sqrt[4]{4}+3\sqrt[4]{324}+2\sqrt[4]{64}$

28) $3\sqrt[4]{2}-2\sqrt[4]{2}-\sqrt[4]{243}$

29) $2\sqrt[4]{6}+2\sqrt[4]{4}+3\sqrt[4]{6}$

30) $-\sqrt[4]{324}+3\sqrt[4]{324}-3\sqrt[4]{4}$

31) $-2\sqrt[4]{243}-\sqrt[4]{96}+2\sqrt[4]{96}$

32) $2\sqrt[4]{2}+2\sqrt[4]{3}+3\sqrt[4]{64}-\sqrt[4]{3}$

33) $2\sqrt[4]{48}-3\sqrt[4]{405}-3\sqrt[4]{48}-\sqrt[4]{162}$

34) $-3\sqrt[5]{6}-\sqrt[5]{64}+2\sqrt[5]{192}-2\sqrt[5]{64}$

ANSWERS to Practice Exercises
Section 3.3: Add and Subtract Radical Expressions

1) $6\sqrt{5}$

2) $-3\sqrt{6}-5\sqrt{3}$

3) $-3\sqrt{2}+6\sqrt{5}$

4) $\sqrt{6}-\sqrt{3}$

5) $-5\sqrt{6}$

6) $4\sqrt{3}$

7) $3\sqrt{6}+5\sqrt{5}$

8) $-\sqrt{5}$

9) $-8\sqrt{2}$

10) $-6\sqrt{6}+9\sqrt{3}$

11) $-3\sqrt{6}+\sqrt{3}$

12) $3\sqrt{5}-8\sqrt{3}$

13) $-2\sqrt{2}$

14) $8\sqrt{5}-\sqrt{3}$

15) $5\sqrt{2}$

16) $9\sqrt{3}$

17) $3\sqrt{2}+3\sqrt{6}$

18) $-12\sqrt{2}+2\sqrt{5}$

19) $3\sqrt{2}$

20) $-4\sqrt{6}+4\sqrt{5}$

21) $-\sqrt{5}-3\sqrt{6}$

22) $8\sqrt{6}-9\sqrt{3}+4\sqrt{2}$

23) $-\sqrt{6}-10\sqrt{3}$

24) $2\sqrt[3]{2}$

25) $6\sqrt[3]{5}-3\sqrt[3]{3}$

26) $-\sqrt[4]{3}$

27) $10\sqrt[4]{4}$

28) $\sqrt[4]{2}-3\sqrt[4]{3}$

29) $5\sqrt[4]{6}+2\sqrt[4]{4}$

30) $3\sqrt[4]{4}$

31) $-6\sqrt[4]{3}+2\sqrt[4]{6}$

32) $2\sqrt[4]{2}+\sqrt[4]{3}+6\sqrt[4]{4}$

33) $-2\sqrt[4]{3}-9\sqrt[4]{5}-3\sqrt[4]{2}$

34) $\sqrt[5]{6}-6\sqrt[5]{2}$

Section 3.4: Multiply and Divide Radical Expressions

Objective: Multiply and divide radical expressions using the product and quotient rules for radicals.

MULTIPLYING RADICAL EXPRESSIONS

The product rule of radicals we used previously can be generalized as follows:

PRODUCT RULE OF RADICALS

For any *nonnegative* real numbers b and d,

$$a\sqrt[n]{b} \cdot c\sqrt[n]{d} = a \cdot c\sqrt[n]{b \cdot d}$$

In words, this rule states that we are allowed to multiply the factors outside the radical and we are allowed to multiply the factors inside the radicals, as long as the indices match.

Example 1. Multiply.

$-5\sqrt{14} \cdot 4\sqrt{6}$	Multiply outside and inside the radical
$= -20\sqrt{84}$	Simplify the radical, divisible by 4
$= -20\sqrt{4 \cdot 21}$	Take the square root where possible
$= -20 \cdot 2\sqrt{21}$	Multiply coefficients
$= -40\sqrt{21}$	Our Answer

The same process works with higher roots.

Example 2. Multiply.

$2\sqrt[3]{18} \cdot 6\sqrt[3]{15}$	Multiply outside and inside the radical
$= 12\sqrt[3]{270}$	Simplify the radical, divisible by 27
$= 12\sqrt[3]{27 \cdot 10}$	Take the square root where possible
$= 12 \cdot 3\sqrt[3]{10}$	Multiply coefficients
$= 36\sqrt[3]{10}$	Our Answer

When multiplying radical expressions we can still use the distributive property or FOIL just as we could when multiplying polynomials.

Example 3. Multiply.

$7\sqrt{6}(3\sqrt{10}-5\sqrt{15})$	Distribute, following rules for multiplying radicals
$=21\sqrt{60}-35\sqrt{90}$	Simplify radicals, finding perfect square factors
$=21\sqrt{4\cdot 15}-35\sqrt{9\cdot 10}$	Take the square root where possible
$=21\cdot 2\sqrt{15}-35\cdot 3\sqrt{10}$	Multiply coefficients
$=42\sqrt{15}-105\sqrt{10}$	Our Answer

Example 4. Multiply.

$(\sqrt{5}-2\sqrt{3})(4\sqrt{10}+6\sqrt{6})$	FOIL, following rules for multiplying radicals
$=4\sqrt{50}+6\sqrt{30}-8\sqrt{30}-12\sqrt{18}$	Simplify radicals, finding perfect square factors
$=4\sqrt{25\cdot 2}+6\sqrt{30}-8\sqrt{30}-12\sqrt{9\cdot 2}$	Take the square root where possible
$=4\cdot 5\sqrt{2}+6\sqrt{30}-8\sqrt{30}-12\cdot 3\sqrt{2}$	Multiply coefficients
$=20\sqrt{2}+6\sqrt{30}-8\sqrt{30}-36\sqrt{2}$	Combine like radicals
$=-16\sqrt{2}-2\sqrt{30}$	Our Answer

Example 5. Multiply.

$(2\sqrt{5}-3\sqrt{6})(7\sqrt{2}-8\sqrt{7})$	FOIL, following rules for multiplying radicals
$=14\sqrt{10}-16\sqrt{35}-21\sqrt{12}+24\sqrt{42}$	Simplify radicals, finding perfect square factors
$=14\sqrt{10}-16\sqrt{35}-21\sqrt{4\cdot 3}+24\sqrt{42}$	Take the square root where possible
$=14\sqrt{10}-16\sqrt{35}-21\cdot 2\sqrt{3}+24\sqrt{42}$	Multiply coefficients
$=14\sqrt{10}-16\sqrt{35}-42\sqrt{3}+24\sqrt{42}$	Our Answer

The next example shows how to use FOIL to square a radical expression with two terms.

Example 6. Multiply.

$(\sqrt{2}+\sqrt{3})^2$	Write as a product
$=(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})$	FOIL, following rules for multiplying radicals
$=\sqrt{4}+\sqrt{6}+\sqrt{6}+\sqrt{9}$	Take the square root where possible
$=2+\sqrt{6}+\sqrt{6}+3$	Combine like terms
$=5+2\sqrt{6}$	Our Answer

As we are multiplying we always look at our final answer to check if all the radicals are simplified and all like radicals have been combined.

DIVIDING RADICAL EXPRESSIONS

Division with radicals is very similar to multiplication. If we think about division as reducing fractions, we can reduce the coefficients outside the radicals and reduce the values inside the radicals to get our final answer.

QUOTIENT RULE OF RADICALS

For any *nonnegative* real numbers b and d,

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}}=\frac{a}{c}\sqrt[n]{\frac{b}{d}}$$

Example 7. Divide.

$\frac{15\sqrt[3]{108}}{20\sqrt[3]{2}}$	Reduce $\frac{15}{20}$ by dividing common factor of 5; reduce $\frac{\sqrt[3]{108}}{\sqrt[3]{2}}$ by dividing 108 by 2
$=\frac{3\sqrt[3]{54}}{4}$	Simplify radical, 54 is divisible by 27
$=\frac{3\sqrt[3]{27\cdot 2}}{4}$	Take the cube root of 27
$=\frac{3\cdot 3\sqrt[3]{2}}{4}$	Multiply coefficients
$=\frac{9\sqrt[3]{2}}{4}$	Our Answer

Example 8. Divide.

$$\frac{\sqrt{50x^3y^7}}{\sqrt{2xy^2}}$$ Divide $\frac{50}{2}, \frac{x^3}{x}$, and $\frac{y^7}{y^2}$

$$=\sqrt{25x^2y^5}$$ Simplify radical, 25 is a perfect square, divide exponents by 2

$$=5xy^2\sqrt{y}$$ Our Answer

There is one catch to dividing radical expressions. It is considered bad practice to have a radical in the denominator of our final answer. We will see how to handle this situation in the next section.

Practice Exercises
Section 3.4: Multiply and Divide Radical Expressions

Perform the indicated operation.

1) $3\sqrt{5}\cdot -4\sqrt{16}$

2) $-5\sqrt{10}\cdot\sqrt{15}$

3) $\sqrt{12m}\cdot\sqrt{15m}$

4) $\sqrt{5r^3}\cdot -5\sqrt{10r^2}$

5) $\sqrt[3]{4x^3}\cdot\sqrt[3]{2x^4}$

6) $\sqrt{3}(4-\sqrt{6})$

7) $\sqrt{6}(\sqrt{2}+2)$

8) $\sqrt{10}(\sqrt{5}+\sqrt{2})$

9) $-5\sqrt{15}(3\sqrt{3}+2)$

10) $\sqrt{7}(\sqrt{3}+5\sqrt{14})$

11) $6\sqrt{10}(5n+\sqrt{2})$

12) $\sqrt{15}(\sqrt{5}-3\sqrt{3v})$

13) $(2+2\sqrt{2})(-3+\sqrt{2})$

14) $(-2+\sqrt{3})(-5+2\sqrt{3})$

15) $(\sqrt{5}-5)(2\sqrt{5}-1)$

16) $(2\sqrt{3}+\sqrt{5})(2\sqrt{3}-\sqrt{5})$

17) $(\sqrt{2a}+2\sqrt{3a})(3\sqrt{2a}+\sqrt{5a})$

18) $(2\sqrt{2p}+\sqrt{q})(3\sqrt{2p}+\sqrt{q})$

19) $(-5-4\sqrt{3})(-3-4\sqrt{3})$

20) $(7-\sqrt{3})^2$

21) $\frac{\sqrt{12}}{5\sqrt{100}}$

22) $\frac{\sqrt{15}}{2\sqrt{4}}$

23) $\frac{\sqrt{5}}{4\sqrt{125}}$

24) $\frac{\sqrt{12}}{\sqrt{3}}$

ANSWERS to Practice Exercises
Section 3.4: Multiply and Divide Radical Expressions

1) $-48\sqrt{5}$
2) $-25\sqrt{6}$
3) $6m\sqrt{5}$
4) $-25r^2\sqrt{2r}$
5) $2x^2\sqrt[3]{x}$
6) $4\sqrt{3}-3\sqrt{2}$
7) $2\sqrt{3}+2\sqrt{6}$
8) $5\sqrt{2}+2\sqrt{5}$
9) $-45\sqrt{5}-10\sqrt{15}$
10) $\sqrt{21}+35\sqrt{2}$
11) $30n\sqrt{10}+12\sqrt{5}$
12) $5\sqrt{3}-9\sqrt{5v}$
13) $-2-4\sqrt{2}$
14) $16-9\sqrt{3}$
15) $15-11\sqrt{5}$
16) 7
17) $6a+a\sqrt{10}+6a\sqrt{6}+2a\sqrt{15}$
18) $12p+5\sqrt{2pq}+q$
19) $63+32\sqrt{3}$
20) $52-14\sqrt{3}$
21) $\frac{\sqrt{3}}{25}$
22) $\frac{\sqrt{15}}{4}$
23) $\frac{1}{20}$
24) 2

Section 3.5: Rationalize Denominators

Objective: Rationalize the denominators of radical expressions.

It is considered bad practice to have a radical in the denominator of a fraction in final form. If there is a radical in the denominator, we will *rationalize* it or clear out any radicals in the denominator.

RATIONALIZING DENOMINATORS CONTAINING ONE TERM

First, we will focus on rationalizing denominators with a single radical term that is a square root in the denominator. Multiply both the numerator and denominator by the same square root to produce a perfect square in the denominator. Use the property for a *nonnegative* number *a:* $(\sqrt{a})(\sqrt{a}) = (\sqrt{a^2}) = a$.

Example 1. Rationalize the denominator.

$\dfrac{\sqrt{6}}{\sqrt{5}}$ Multiply numerator and denominator by $\sqrt{5}$

$= \dfrac{\sqrt{6}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}}$ Multiply the numerators; multiply $\sqrt{5} \cdot \sqrt{5} = \sqrt{25} = 5$ in the denominator

$= \dfrac{\sqrt{30}}{5}$ Our Answer

Example 2. Rationalize the denominator.

$\dfrac{12}{\sqrt{3x}}$ Multiply numerator and denominator by $\sqrt{3x}$

$= \dfrac{12}{\sqrt{3x}} \cdot \dfrac{\sqrt{3x}}{\sqrt{3x}}$ Multiply the numerators; multiply $\sqrt{3x} \cdot \sqrt{3x} = \sqrt{9x^2} = 3x$ in the denominator

$= \dfrac{12\sqrt{3x}}{3x}$ Reduce the fraction

$= \dfrac{4\sqrt{3x}}{x}$ Our Answer

Example 3. Rationalize the denominator.

$\dfrac{2+\sqrt{3}}{\sqrt{7}}$	Multiply numerator and denominator by $\sqrt{7}$
$=\dfrac{(2+\sqrt{3})}{\sqrt{7}}\cdot\dfrac{\sqrt{7}}{\sqrt{7}}$	Distribute in the numerator; multiply $\sqrt{7}\cdot\sqrt{7}=\sqrt{49}=7$ in the denominator
$=\dfrac{2\sqrt{7}+\sqrt{21}}{7}$	Our Answer

Example 4. Rationalize the denominator.

$\dfrac{\sqrt{3}-9}{2\sqrt{6}}$	Multiply numerator and denominator by $\sqrt{6}$
$=\dfrac{(\sqrt{3})-9}{2\sqrt{6}}\cdot\dfrac{\sqrt{6}}{\sqrt{6}}$	Distribute in the numerator; multiply $\sqrt{6}\cdot\sqrt{6}=\sqrt{36}=6$ in the denominator
$=\dfrac{\sqrt{18}-9\sqrt{6}}{2\cdot 6}$	Simplify radicals in numerator*;* multiply denominator
$=\dfrac{\sqrt{9\cdot 2}-9\sqrt{6}}{12}$	Take square root where possible
$=\dfrac{3\sqrt{2}-9\sqrt{6}}{12}$	Factor numerator
$=\dfrac{3\left(\sqrt{2}-3\sqrt{6}\right)}{12}$	Reduce by dividing common factor of 3
$=\dfrac{\sqrt{2}-3\sqrt{6}}{4}$	Our Answer

It is important to remember that when reducing the fraction, we cannot reduce with just the 3 and 12 or just the 9 and 12. When we have addition or subtraction in the numerator or denominator, factor and then divide out any common factors.

As we rationalize denominators, it will always be important to constantly check our answer to see if it can be simplified more. We ask ourselves, can the fraction be reduced? Can the radicals be simplified? These steps may happen several times on our way to the solution.

RATIONALIZING DENOMINATORS CONTAINING TWO TERMS

Now we will focus on rationalizing denominators containing two terms with one or more square roots. We will use a different strategy to rationalize the denominator than we did when the denominator had one radical term.

Consider $\frac{2}{\sqrt{3}-5}$. If we were to multiply the denominator by $\sqrt{3}$, we would distribute and end up with $3-5\sqrt{3}$. We have not cleared the radical from the denominator so our current method will not work.

Instead, we will multiply numerator and denominator by the **conjugate** of the denominator. The conjugate has the same terms but with the opposite sign in the middle. In our example with $\sqrt{3}-5$ in the denominator, its conjugate is $\sqrt{3}+5$. When we multiply the conjugates, we get:

$$(\sqrt{3}-5)(\sqrt{3}+5)=3+5\sqrt{3}-5\sqrt{3}-25=3-25=-22$$

When multiplying conjugates, we will no longer have a radical in the denominator.

Example 5. Rationalize the denominator.

$\frac{2}{\sqrt{3}-5}$	Multiply numerator and denominator by $\sqrt{3}+5$, the conjugate of the denominator.
$=\frac{2}{\sqrt{3}-5}\left(\frac{\sqrt{3}+5}{\sqrt{3}+5}\right)$	Distribute in the numerator; multiply conjugates in the denominator.
$=\frac{2\sqrt{3}+10}{3+5\sqrt{3}-5\sqrt{3}-25}$	Simplify the denominator.
$=\frac{2\sqrt{3}+10}{-22}$	Factor numerator using a GCF of -2
$=\frac{-2\left(-\sqrt{3}-5\right)}{-22}$	Reduce by dividing common factor of -2
$=\frac{-\sqrt{3}-5}{11}$	Our Answer

Example 6. Rationalize the denominator.

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}$$

Multiply numerator and denominator by $\sqrt{5}-\sqrt{3}$, the conjugate of the denominator.

$$=\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)$$

Distribute in the numerator;
multiply conjugates in the denominator.

$$=\frac{\sqrt{75}-\sqrt{45}}{5-\sqrt{15}+\sqrt{15}-3}$$

Simplify the radicals in the numerator;
simplify the denominator.

$$=\frac{\sqrt{25}\sqrt{3}-\sqrt{9}\sqrt{5}}{2}$$

Take square roots.

$$=\frac{5\sqrt{3}-3\sqrt{5}}{2}$$

Our Answer

Example 7. Rationalize the denominator.

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}}$$

Multiply numerator and denominator by $4+\sqrt{5x^3}$, the conjugate of the denominator.

$$=\frac{2\sqrt{3x}}{4-\sqrt{5x^3}}\left(\frac{4+\sqrt{5x^3}}{4+\sqrt{5x^3}}\right)$$

Distribute in the numerator;
multiply conjugates in the denominator.

$$=\frac{8\sqrt{3x}+2\sqrt{15x^4}}{16+4\sqrt{5x^3}-4\sqrt{5x^3}-5x^3}$$

Simplify the radicals in the numerator;
simplify the denominator.

$$=\frac{8\sqrt{3x}+2x^2\sqrt{15}}{16-5x^3}$$

Our Answer

Example 8. Rationalize the denominator.

$$\frac{3-\sqrt{5}}{2-\sqrt{3}}$$

Multiply numerator and denominator by $2+\sqrt{3}$, the conjugate of the denominator.

$$=\frac{3-\sqrt{5}}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)$$

Distribute in the numerator; multiply conjugates in the denominator.

$$=\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4+2\sqrt{3}-2\sqrt{3}-3}$$

Simplify the radicals in the numerator; simplify the denominator.

$$=\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{1}$$

Divide each term by 1.

$$=6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}$$

Our Answer

Practice Exercises
Section 3.5: Rationalize Denominators

Rationalize the denominator.

1) $\dfrac{1}{\sqrt{2}}$

2) $\dfrac{5}{\sqrt{3}}$

3) $\dfrac{\sqrt{10}}{\sqrt{6}}$

4) $\dfrac{\sqrt{2}}{3\sqrt{5}}$

5) $\dfrac{2\sqrt{4}}{3\sqrt{3}}$

6) $\dfrac{4\sqrt{3}}{\sqrt{15}}$

7) $\dfrac{\sqrt{5}}{2-\sqrt{3}}$

8) $\dfrac{7}{\sqrt{3}+8}$

9) $\dfrac{2}{\sqrt{7}+\sqrt{2}}$

10) $\dfrac{2\sqrt{3}}{9+3\sqrt{11}}$

11) $\dfrac{1+\sqrt{3}}{5+\sqrt{2}}$

12) $\dfrac{6\sqrt{3}}{\sqrt{5}-3}$

13) $\dfrac{8}{\sqrt{13}-\sqrt{7}}$

14) $\dfrac{4}{2\sqrt{7}-2}$

15) $\dfrac{5x^2}{4\sqrt{3x^2y^3}}$

16) $\dfrac{4}{5\sqrt{3xy^4}}$

17) $\dfrac{\sqrt{2p^2}}{\sqrt{3p}}$

18) $\dfrac{\sqrt{8n^2}}{\sqrt{10n}}$

19) $\dfrac{2-5\sqrt{5}}{4\sqrt{13}}$

20) $\dfrac{\sqrt{5}+4}{4\sqrt{17}}$

21) $\dfrac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$

22) $\dfrac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$

ANSWERS to Practice Exercises
Section 3.5: Rationalize Denominators

1) $\dfrac{\sqrt{2}}{2}$

2) $\dfrac{5\sqrt{3}}{3}$

3) $\dfrac{\sqrt{15}}{3}$

4) $\dfrac{\sqrt{10}}{15}$

5) $\dfrac{4\sqrt{3}}{9}$

6) $\dfrac{4\sqrt{5}}{5}$

7) $2\sqrt{5}+\sqrt{15}$

8) $\dfrac{56-7\sqrt{3}}{61}$

9) $\dfrac{2\left(\sqrt{7}-\sqrt{2}\right)}{5}$

10) $\dfrac{\sqrt{33}-3\sqrt{3}}{3}$

11) $\dfrac{5-\sqrt{2}+5\sqrt{3}-\sqrt{6}}{23}$

12) $\dfrac{-3\left(\sqrt{15}+3\sqrt{3}\right)}{2}$

13) $\dfrac{4\left(\sqrt{13}+\sqrt{7}\right)}{3}$

14) $\dfrac{1+\sqrt{7}}{3}$

15) $\dfrac{5x\sqrt{3y}}{12y^2}$

16) $\dfrac{4\sqrt{3x}}{15xy^2}$

17) $\dfrac{\sqrt{6p}}{3}$

18) $\dfrac{2\sqrt{5n}}{5}$

19) $\dfrac{2\sqrt{13}-5\sqrt{65}}{52}$

20) $\dfrac{\sqrt{85}+4\sqrt{17}}{68}$

21) $\dfrac{\sqrt{6}-9}{3}$

22) $\dfrac{\sqrt{30}-2\sqrt{3}}{18}$

Section 3.6: Rational Exponents

Objectives: Convert between radical notation and exponential notation.
Simplify expressions with rational exponents using the properties of exponents.
Multiply and divide radical expressions with different indices.

We define rational exponents as follows:

DEFINITION OF RATIONAL EXPONENTS:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \text{ and } a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

The denominator of a rational exponent is the same as the index of our radical while the numerator serves as an exponent.

Either form of the definition can be used but we typically use the first form as it will involve smaller numbers.

Notice when the numerator of the exponent is 1, the special case of n^{th} roots follows from the definition:

$$a^{\frac{1}{n}} = (\sqrt[n]{a})^1 = \sqrt[n]{a}$$

CONVERTING BETWEEN EXPONENTIAL AND RADICAL NOTATION

We can use this definition to change any radical expression into an exponential expression.

Example 1. Rewrite with rational exponents.

$(\sqrt[5]{x})^3 = x^{\frac{3}{5}}$	$(\sqrt[6]{3x})^5 = (3x)^{\frac{5}{6}}$	Index is denominator, exponent is numerator
$\dfrac{1}{(\sqrt[7]{a})^3} = a^{-\frac{3}{7}}$	$\dfrac{1}{(\sqrt[3]{xy})^2} = (xy)^{-\frac{2}{3}}$	Negative exponents from reciprocals

We can also change any rational exponent into a radical expression by using the denominator as the index.

Example 2. Rewrite using radical notation.

$a^{\frac{5}{3}} = (\sqrt[3]{a})^5$	$(2mn)^{\frac{2}{7}} = (\sqrt[7]{2mn})^2$
$x^{-\frac{4}{5}} = \dfrac{1}{(\sqrt[5]{x})^4}$	$(xy)^{-\frac{2}{9}} = \dfrac{1}{(\sqrt[9]{xy})^2}$

Exponent is numerator; index is denominator

Negative exponent means reciprocals

The ability to change between exponential expressions and radical expressions allows us to evaluate expressions we had no means of evaluating previously.

Example 3. Use radical notation to rewrite and evaluate.

$16^{\frac{3}{2}}$ Change to radical format;
numerator is exponent, denominator is index

$= (\sqrt{16})^3$ Evaluate radical

$= (4)^3$ Evaluate exponent

$= 64$ Our Answer

Example 4. Use radical notation to rewrite and evaluate.

$27^{-\frac{4}{3}}$ Negative exponent is reciprocal

$= \dfrac{1}{27^{\frac{4}{3}}}$ Change to radical format;
numerator is exponent, denominator is index

$= \dfrac{1}{(\sqrt[3]{27})^4}$ Evaluate radical

$= \dfrac{1}{(3)^4}$ Evaluate exponent

$= \dfrac{1}{81}$ Our Answer

SIMPLIFY EXPRESSIONS WITH RATIONAL EXPONENTS

The largest advantage of being able to change a radical expression into an exponential expression is we are now allowed to use all our exponent properties to simplify. The following table reviews all of our exponent properties.

PROPERTIES OF EXPONENTS

$a^m a^n = a^{m+n}$	$(ab)^m = a^m b^m$	$a^{-m} = \frac{1}{a^m}$
$\frac{a^m}{a^n} = a^{m-n}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\frac{1}{a^{-m}} = a^m$
$(a^m)^n = a^{mn}$	$a^0 = 1$	$\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$

When adding and subtracting with fractions we need to have a common denominator. When multiplying we only need to multiply the numerators together and denominators together. The following examples show several different problems, using different properties to simplify the rational exponents.

Example 5. Simplify.

$a^{\frac{2}{3}}b^{\frac{1}{2}}a^{\frac{1}{6}}b^{\frac{1}{5}}$ Need common denominator for a s (6) and for b s (10)

$= a^{\frac{4}{6}}b^{\frac{5}{10}}a^{\frac{1}{6}}b^{\frac{2}{10}}$ Add exponents on a s and b s

$= a^{\frac{5}{6}}b^{\frac{7}{10}}$ Our Answer

Example 6. Simplify.

$\left(x^{\frac{1}{3}}y^{\frac{2}{5}}\right)^{\frac{3}{4}}$ Multiply each exponent by $\frac{3}{4}$; reduce fractions

$= x^{\frac{1}{4}}y^{\frac{3}{10}}$ Our Answer

Example 7. Simplify.

$\dfrac{x^2 y^{\frac{2}{3}}}{x^{\frac{7}{2}} y^0}$	Need common denominator for xs (2) to subtract exponents
$=\dfrac{x^{\frac{4}{2}} y^{\frac{2}{3}}}{x^{\frac{7}{2}} y^0}$	Subtract exponents on x in denominator, $y^0 = 1$
$= x^{-\frac{3}{2}} y^{\frac{2}{3}}$	Negative exponent moves down to denominator
$=\dfrac{y^{\frac{2}{3}}}{x^{\frac{3}{2}}}$	Our Answer

MULTIPLY AND DIVIDE RADICAL EXPRESSIONS WITH DIFFERENT INDICES

We will use rational exponents to multiply or divide radical expressions having different indices. We will convert each radical expression to its equivalent exponential expression. Then, we will apply the appropriate exponent property. For our answer, we will convert the exponential expression to its equivalent radical expression. Our answer will then be written as a single radical expression.

Example 8. Multiply, writing the expression using a single radical.

$\sqrt[5]{x} \cdot \sqrt{x}$	Rewrite radical expressions using rational exponents
$= x^{\frac{1}{5}} \cdot x^{\frac{1}{2}}$	Need common denominator of 10 to add exponents
$= x^{\frac{2}{10}} \cdot x^{\frac{5}{10}}$	Add exponents
$= x^{\frac{7}{10}}$	Rewrite as a radical expression
$= \sqrt[10]{x^7}$	Our Answer

Example 9. Divide, writing the expression using a single radical.

$\dfrac{\sqrt[3]{y^2}}{\sqrt[5]{y^2}}$	Rewrite radical expressions using rational exponents
$=\dfrac{y^{\frac{2}{3}}}{y^{\frac{2}{5}}}$	Need common denominator of 15 to subtract exponents
$=\dfrac{y^{\frac{10}{15}}}{y^{\frac{6}{15}}}$	Subtract exponents
$=y^{\frac{4}{15}}$	Rewrite as a radical expression
$=\sqrt[15]{y^4}$	Our Answer

It is important to remember that as we simplify with rational exponents, we are using the exact same properties we used when simplifying integer exponents. The only difference is we need to follow our rules for fractions as well. It may be worth reviewing your notes on exponent properties to be sure you are comfortable with using the properties.

Practice Exercises
Section 3.6: Rational Exponents

Write each expression in radical form.

1) $m^{\frac{3}{5}}$

2) $(10r)^{-\frac{3}{4}}$

3) $(7x)^{\frac{3}{2}}$

4) $(6b)^{-\frac{3}{4}}$

Write each expression in exponential form.

5) $\dfrac{1}{\left(\sqrt{6x}\right)^3}$

6) $\sqrt{v}$

7) $\dfrac{1}{(\sqrt[4]{n})^7}$

8) $\sqrt{5a}$

Evaluate.

9) $8^{\frac{2}{3}}$

10) $16^{\frac{1}{4}}$

11) $4^{\frac{3}{2}}$

12) $100^{-\frac{3}{2}}$

13) $27^{-\frac{1}{3}}$

14) $32^{\frac{3}{5}}$

15) $81^{-\frac{3}{4}}$

16) $25^{\frac{3}{2}}$

Simplify. Your answer should contain only positive exponents.

17) $x^{\frac{1}{3}}y \cdot xy^{\frac{3}{2}}$

18) $4v^{\frac{2}{3}} \cdot v^{-1}$

19) $\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^{-1}$

20) $\left(x^{\frac{5}{3}}y^{-2}\right)^0$

21) $\left(x^0y^{\frac{1}{3}}\right)^{\frac{3}{2}}x^0$

22) $u^{-\frac{5}{4}}v^2 \cdot \left(u^{\frac{3}{2}}\right)^{-\frac{3}{2}}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 3.6 (continued)

Simplify. Your answer should contain only positive exponents.

23) $\dfrac{a^{\frac{3}{4}}b^{-1}\cdot b^{\frac{7}{4}}}{3b^{-1}}$

24) $\dfrac{2x^{-2}y^{\frac{5}{3}}}{x^{-\frac{5}{4}}y^{-\frac{5}{3}}\cdot xy^{\frac{1}{2}}}$

25) $\dfrac{3y^{-\frac{5}{4}}}{y^{-1}\cdot 2y^{-\frac{1}{3}}}$

26) $\dfrac{ab^{\frac{1}{3}}\cdot 2b^{-\frac{5}{4}}}{4a^{-\frac{1}{2}}b^{\frac{2}{3}}}$

27) $\left(\dfrac{m^{\frac{3}{2}}n^{-2}}{\left(mn^{\frac{4}{3}}\right)^{-1}}\right)^{\frac{7}{4}}$

28) $\dfrac{\left(y^{-\frac{1}{2}}\right)^{\frac{3}{2}}}{x^{\frac{3}{2}}y^{\frac{1}{2}}}$

29) $\dfrac{\left(m^{2}n^{\frac{1}{2}}\right)^{0}}{n^{\frac{3}{4}}}$

30) $\dfrac{y^{0}}{\left(x^{\frac{3}{4}}y^{-1}\right)^{\frac{1}{3}}}$

31) $\dfrac{\left(x^{-\frac{4}{3}}y^{-\frac{1}{3}}\cdot y\right)^{-1}}{x^{\frac{1}{3}}y^{-2}}$

32) $\dfrac{\left(x^{\frac{1}{2}}y^{0}\right)^{-\frac{4}{3}}}{y^{4}\cdot x^{-2}y^{-\frac{2}{3}}}$

Perform the indicated operation, writing the expression using a single radical.

33) $\sqrt{x}\cdot\sqrt[4]{x}$

34) $\dfrac{\sqrt[5]{x^{2}}}{\sqrt[6]{x}}$

ANSWERS to Practice Exercises
Section 3.6: Rational Exponents

1) $\left(\sqrt[5]{m}\right)^3$

2) $\dfrac{1}{\left(\sqrt[4]{10r}\right)^3}$

3) $\left(\sqrt{7x}\right)^3$

4) $\dfrac{1}{\left(\sqrt[4]{6b}\right)^3}$

---

5) $(6x)^{-\frac{3}{2}}$

6) $v^{\frac{1}{2}}$

7) $n^{-\frac{7}{4}}$

8) $(5a)^{\frac{1}{2}}$

---

9) 4

10) 2

11) 8

12) $\dfrac{1}{1000}$

13) $\dfrac{1}{3}$

14) 8

15) $\dfrac{1}{27}$

16) 125

---

17) $x^{\frac{4}{3}}y^{\frac{5}{2}}$

18) $\dfrac{4}{v^{\frac{1}{3}}}$

19) $\dfrac{1}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

20) 1

21) $y^{\frac{1}{2}}$

22) $\dfrac{v^2}{u^{\frac{7}{2}}}$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 3.6 (continued)

23) $\dfrac{a^{\frac{3}{4}}b^{\frac{7}{4}}}{3}$

24) $\dfrac{2y^{\frac{17}{6}}}{x^{\frac{7}{4}}}$

25) $\dfrac{3y^{\frac{1}{12}}}{2}$

26) $\dfrac{a^{\frac{3}{2}}}{2b^{\frac{19}{12}}}$

27) $\dfrac{m^{\frac{35}{8}}}{n^{\frac{7}{6}}}$

28) $\dfrac{1}{x^{\frac{3}{2}}y^{\frac{5}{4}}}$

29) $\dfrac{1}{n^{\frac{3}{4}}}$

30) $\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{4}}}$

31) $xy^{\frac{4}{3}}$

32) $\dfrac{x^{\frac{4}{3}}}{y^{\frac{10}{3}}}$

---

33) $\sqrt[4]{x^3}$

34) $\sqrt[30]{x^7}$

Section 3.7: Solving Radical Equations

Objective: Solve equations with radicals and check for extraneous solutions.

In this section, we solve equations that have roots in the problem. As you might expect, to clear a root we can raise both sides to an exponent. Thus, to clear a square root, we can raise both sides to the second power. To clear a cube root, we can raise both sides to the third power.

There is one catch to solving radical equations. Sometimes we end up with proposed solutions that do not actually work in the original equation. This will only happen if the index on the root is even, and it will not happen all the time for those roots. So, for radical equations solved by raising both sides to an *even* power, we must check our answers by substituting each result into the original equation. If a proposed solution does not work, it is called an *extraneous* solution, and is not included in the final solution.

NOTE: When solving a radical equation with an *even* index, always check your answers!

Example 1. Solve the equation.

$\sqrt{7x+2}=4$ Even index; we will have to check all results

$\left(\sqrt{7x+2}\right)^2=4^2$ Square both sides, simplify exponents

$7x+2=16$ Solve

$\underline{-2\quad -2}$ Subtract 2 from both sides

$\dfrac{7x}{7}=\dfrac{14}{7}$ Divide both sides by 7.

$x=2$ Need to check this result in the original equation

$\sqrt{7(2)+2}=4$ Multiply

$\sqrt{14+2}=4$ Add

$\sqrt{16}=4$ Square root

$4=4$ True, it works

$x=2$ Our Solution

Example 2. Solve the equation.

$\sqrt[3]{x-1}=-4$	Odd index; we don't need to check our results
$\left(\sqrt[3]{(x-1)}\right)^3=(-4)^3$	Cube both sides, simplify exponents
$x-1=-64$	Solve
$\underline{\quad +1 \quad +1}$	Add 1 to both sides
$x=-63$	Our Solution

Example 3. Solve the equation.

$\sqrt[4]{3x+6}=-3$	Even index; we will have to check all results
$\left(\sqrt[4]{3x+6}\right)^4=(-3)^4$	Raise both sides to the fourth power
$3x+6=81$	Solve
$\underline{\quad -6 \quad -6}$	Subtract 6 from both sides
$\frac{3x}{3}=\frac{75}{3}$	Divide both sides by 3
$x=25$	Need to check this result in the original equation
$\sqrt[4]{3(25)+6}=-3$	Multiply
$\sqrt[4]{75+6}=-3$	Add
$\sqrt[4]{81}=-3$	Simplify the radical
$3=-3$	False, extraneous solution; thus, $x=25$ is not a solution
No Solution	Our Solution

If the radical is not alone on one side of the equation, we will have to isolate the radical before we raise it to an exponent.

Example 4. Solve the equation.

$x+\sqrt{4x+1}=5$	Even index, we will have to check all results
$\underline{-x \qquad\qquad -x}$	Isolate radical by subtracting *x* from both sides
$\sqrt{4x+1}=5-x$	Square both sides
$\left(\sqrt{4x+1}\right)^2=(5-x)^2$	Evaluate exponents, recall $(a-b)^2=a^2-2ab+b^2$

$4x+1=25-10x+x^2$ $\underline{-4x-1 \quad\quad -4x-1}$	Rewrite equation equal to zero Subtract $4x$ and 1 from both sides; reorder terms
$0=x^2-14x+24$	Factor
$0=(x-12)(x-2)$	Set each factor equal to zero
$x-12=0$ or $x-2=0$ $\underline{+12 \quad +12}$ $\quad$ $\underline{+2 \quad +2}$	Solve each equation
$x=12$ or $x=2$	Need to check both results by substituting each into the original equation
$(12)+\sqrt{4(12)+1}=5$	Check $x=12$ first; multiply inside the radical
$12+\sqrt{48+1}=5$	Add inside the root sign
$12+\sqrt{49}=5$	Take the square root
$12+7=5$	Add
$19=5$	False, extraneous solution; thus $x=12$ is not a solution
$(2)+\sqrt{4(2)+1}=5$	Check $x=2$ second; multiply inside the radical
$2+\sqrt{8+1}=5$	Add inside the root sign
$2+\sqrt{9}=5$	Take the square root
$2+3=5$	Add
$5=5$	True, it works
$x=2$	Our Solution

The above example illustrates that as we square both sides of the equation we could end up with a quadratic equation. In this case, we must set the equation to zero and solve by factoring. We will have to check both solutions if the root in the problem was even (for example, a square root or a fourth root). Sometimes both values work, sometimes only one value works, and sometimes neither value works.

If there is more than one square root in a problem we will clear all the roots at the same time. This means we must first make sure that one root is isolated on one side of the equal sign before squaring both sides.

Example 5. Solve the equation.

$\sqrt{3x-8}-\sqrt{x}=0$	Even index, we will have to check all results
$\underline{\quad +\sqrt{x} \quad +\sqrt{x}}$	Isolate first root by adding $\sqrt{x}$ to both sides
$\sqrt{3x-8}=\sqrt{x}$	Square both sides
$\left(\sqrt{3x-8}\right)^2=\left(\sqrt{x}\right)^2$	Evaluate exponents
$3x-8=x$	Solve the equation
$\underline{-3x \quad\quad -3x}$	Subtract $3x$ from both sides
$\frac{-8}{-2}=\frac{-2x}{-2}$	Divide both sides by -2
$4=x$	Need to check result in original equation
$\sqrt{3(4)-8}-\sqrt{4}=0$	Multiply inside the root sign
$\sqrt{12-8}-\sqrt{4}=0$	Subtract inside the root sign
$\sqrt{4}-\sqrt{4}=0$	Take roots
$2-2=0$	Subtract
$0=0$	True, it works
$x=4$	Our Solution

When the index of the roots is not 2, we need to raise both sides of the equation to the power that corresponds to that index.

Example 6. Solve the equation.

$\sqrt[4]{x-1}=\sqrt[4]{8}$	Even index, we will have to check all results
$\left(\sqrt[4]{x-1}\right)^4=\left(\sqrt[4]{8}\right)^4$	Raise both sides to the fourth power
$x-1=8$	Evaluate exponents
$\underline{\quad +1 \quad +1}$	Add 1 to both sides of the equation
$x=9$	Need to check result in original equation
$\sqrt[4]{(9)-1}=\sqrt[4]{8}$	Subtract
$\sqrt[4]{8}=\sqrt[4]{8}$	True, it works
$x=9$	Our Solution

Example 7. Solve the equation.

$\sqrt[3]{x^2+5}=\sqrt[3]{x^2-4x+1}$	Odd index, we don't need to check our results
$\left(\sqrt[3]{x^2+5}\right)=\left(\sqrt[3]{x^2-4x+1}\right)^3$	Raise both sides to the third power
$\begin{array}{r} x^2+5=x^2-4x+1 \\ \underline{-x^2 \qquad -x^2} \end{array}$	Subtract x^2 on both sides of the equation
$\begin{array}{r} 5=-4x+1 \\ \underline{-1 \qquad\quad -1} \end{array}$	Subtract 1 from both sides of the equation
$\dfrac{4}{-4}=\dfrac{-4x}{-4}$	Divide both sides by -4
$-1=x$	
$x=-1$	Our Solution

Practice Exercises
Section 3.7: Solving Radical Equations

Solve.

1) $\sqrt{2x+3}-3=0$

2) $\sqrt{5x+1}-4=0$

3) $\sqrt{6x-5}-x=0$

4) $\sqrt{x+2}-\sqrt{3x}=0$

5) $3+x=\sqrt{6x+13}$

6) $x-1=\sqrt{7-x}$

7) $\sqrt{3-3x}-1=2x$

8) $\sqrt{2x+2}=\sqrt{5x-1}$

9) $\sqrt{4x+5}-\sqrt{x+4}=0$

10) $\sqrt{3x+4}-\sqrt{x+2}=0$

11) $\sqrt{2x-4}-\sqrt{x+3}=0$

12) $\sqrt[3]{3x+1}=-2$

13) $\sqrt[4]{x-3}=2$

14) $\sqrt[4]{7x-5}=-2$

15) $\sqrt[5]{6x-2}=-2$

16) $\sqrt[3]{2x-1}=\sqrt[3]{7x+9}$

ANSWERS to Practice Exercises
Section 3.7: Solving Radical Equations

1) 3

2) 3

3) $1, 5$

4) 1

5) ± 2

6) 3

7) $\frac{1}{4}$

8) 1

9) $-\frac{1}{3}$

10) -1

11) 7

12) -3

13) 19

14) no solution

15) -5

16) -2

Section 3.8: Complex Numbers

Objective: Add, subtract, multiply, divide, and simplify expressions using complex numbers.

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, mathematicians create new ways for dealing with the problem. Throughout history this has been the case with the need for a number that is nothing (0), smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for the square roots of negative numbers. To work with the square root of negative numbers and to solve equations such as $x^2 = -1$; mathematicians have created a new number system called the complex numbers. First, we define the imaginary unit:

DEFINITION OF THE IMAGINARY UNIT *i*

$$i = \sqrt{-1} \quad \text{where } i^2 = -1$$

With this definition, the square root of a negative number can now be expressed as a multiple of i. We will use the product rule of radicals and simplify the negative as a factor of negative one. This process is shown in the following examples.

Example 1. Write in terms of i.

$\sqrt{-16}$	Consider the negative as a factor of -1
$=\sqrt{-1 \cdot 16}$	Take each root, square root of -1 is i
$= 4i$	Our Answer

Example 2. Write in terms of i.

$\sqrt{-24}$	Find perfect square factors, including -1
$=\sqrt{-1 \cdot 4 \cdot 6}$	Square root of -1 is i, square root of 4 is 2
$= 2i\sqrt{6}$	Our Answer

Then, mathematicians created a new number system called the set of complex numbers. A **complex number** is one that contains both a real and imaginary part.

DEFINITION OF COMPLEX NUMBERS

$$a+bi$$

where a and b are real numbers

We call a the real part and b the imaginary part. Examples of complex numbers include $2+5i, -3+i\sqrt{5}, -6i$ because $-6i = 0-6i$ *;* and 3 because $3 = 3+0i$.

ADDING AND SUBTRACTING COMPLEX NUMBERS

The operations of addition, subtraction, and multiplication of complex numbers are performed very similarly to how they are done with polynomials. A new technique will be needed for division though.

When adding and subtracting complex numbers, we combine like terms by adding or subtracting the real parts, adding or subtracting the imaginary parts, and expressing the answer in the form of a complex number $a+bi$.

Example 3. Add.

$(2+5i)+(4-7i)$	Combine real parts, $2+4$ imaginary parts $5i-7i$
$=6-2i$	Our Answer

It is important to notice what operation we are doing. Students often see the parentheses and think that means FOIL. We only use FOIL to multiply. This problem is an addition problem, so we simply add the real and imaginary parts.

For subtraction of complex numbers, the idea is the same, but we need to remember to first distribute the negative onto all the terms in the parentheses.

Example 4. Subtract.

$(4-8i)-(3-5i)$	Distribute the negative
$=4-8i-3+5i$	Combine real parts, $4-3$ imaginary parts $-8i+5i$
$=1-3i$	Our Answer

Addition and subtraction can appear together in one problem.

Example 5. Perform the indicated operations.

$(5i)-(3+8i)+(-4+7i)$	Distribute the negative
$=5i-3-8i-4+7i$	Combine real parts, $-3-4$ imaginary parts $5i-8i+7i$
$=-7+4i$	Our Answer

MULTIPLYING COMPLEX NUMBERS

Multiplying complex numbers is the same as multiplying polynomials, but we replace i^2 with -1.

Example 6. Multiply.

$(3i)(7i)$	Multiply
$=21i^2$	Replace i^2 with -1
$=21(-1)$	Multiply
$=-21$	Our Answer

When multiplying complex radicals, it is important that we first rewrite as multiples of i.

Example 7. Multiply.

$\sqrt{-6}\sqrt{-3}$	Simplify each root using $i=\sqrt{-1}$
$=(i\sqrt{6})(i\sqrt{3})$	Multiply
$=i^2\sqrt{18}$	Replace i^2 with -1
$=-\sqrt{18}$	Simplify the radical
$=-\sqrt{9\cdot 2}$	Take square root of 9
$-3\sqrt{2}$	Our Answer

Example 8. Multiply.

$5i(3i-7)$	Distribute
$=15i^2-35i$	Replace i^2 with -1
$=15(-1)-35i$	Multiply
$=-15-35i$	Our Answer

Example 9. Multiply.

$(2-4i)(3+5i)$	FOIL
$=6+10i-12i-20i^2$	Replace i^2 with -1
$=6+10i-12i-20(-1)$	Multiply
$=6+10i-12i+20$	Combine real parts, $6+20$ imaginary parts $10i-12i$
$=26-2i$	Our Answer

Remember when squaring a binomial, we write as a product of two same binomials and then FOIL.

Example 10. Multiply.

$(4-5i)^2$	Write as a product of two same complex numbers
$=(4-5i)(4-5i)$	FOIL
$=16-20i-20i+25i^2$	Replace i^2 with -1
$=16-20i-20i+25(-1)$	Multiply
$=16-20i-20i-25$	Combine real parts, $16-25$ imaginary parts $-20i-20i$
$=-9-40i$	Our Answer

Example 11. Multiply.

$(2+3i)(2-3i)$	FOIL
$=4-6i+6i-9i^2$	Replace i^2 with -1
$=4-6i+6i-9(-1)$	Multiply
$=4-6i+6i+9$	Combine real parts, $4+9$ imaginary parts $-6i+6i$
$=13$	Our Answer

Notice how the product of the two complex numbers above resulted in a real number.

The complex numbers $a+bi$ and $a-bi$ are called **complex conjugates** of each other. Notice that $(a+bi)\,(a-bi)=a^2+b^2$. When we multiply complex conjugates, the result is always a real number.

DIVIDING COMPLEX NUMBERS

Dividing complex numbers also has one thing we need to be careful of. If i is $\sqrt{-1}$, and it is in the denominator of a fraction, then we have a radical in the denominator! This means we will want to rationalize our denominator so there are no i s. This is done by multiplying numerator and denominator by the conjugate of the denominator.

Example 12. Divide.

$\dfrac{2-6i}{4+8i}$	Multiply by conjugate of denominator, $4-8i$
$=\dfrac{(2-6i)}{(4+8i)}\cdot\dfrac{(4-8i)}{(4-8i)}$	FOIL in numerator, denominator is difference of squares
$=\dfrac{8-16i-24i+48i^2}{16-64(-1)}$	Replace i^2 with -1
$=\dfrac{8-16i-24i+48(-1)}{16-64(-1)}$	Multiply
$=\dfrac{8-16i-24i-48}{16+64}$	Combine real and imaginary parts
$=\dfrac{-40-40i}{80}$	Reduce, dividing each term by 80
$=-\dfrac{40}{80}-\dfrac{40i}{80}$	Reduce and write in the form of a complex number
$=-\dfrac{1}{2}-\dfrac{1}{2}i$	Our Answer

Example 13. Divide.

$=\dfrac{7+3i}{-5i}$	Multiply by conjugate of denominator, $5i$
$=\dfrac{(7+3i)}{-5i}\cdot\dfrac{5i}{5i}$	Distribute $5i$ in numerator
$=\dfrac{35i+15i^2}{-25i^2}$	Replace i^2 with -1

$=\dfrac{35i+15(-1)}{-25(-1)}$ Multiply

$=\dfrac{35i-15}{25}$ Reduce, dividing each term by 25

$=-\dfrac{15}{25}+\dfrac{35}{25}i$ Reduce and write in the form of a complex number

$=-\dfrac{3}{5}+\dfrac{7}{5}i$ Our Answer

Practice Exercises
Section 3.8: Complex Numbers

Write in terms of i.

1) $\sqrt{-81}$

2) $\sqrt{-45}$

Multiply.

3) $\sqrt{-4}\cdot\sqrt{-9}$

4) $\sqrt{-10}\cdot\sqrt{-2}$

5) $\sqrt{-12}\cdot\sqrt{-2}$

6) $\sqrt{-3}\cdot\sqrt{27}$

Perform the indicated operation, writing the answer in the form of a complex number $a+bi$.

7) $3-(-8+4i)$

8) $(-8i)-(7i)-(5-3i)$

9) $(7i)-(3-2i)$

10) $(-4-i)+(1-5i)$

11) $(-6i)-(3+7i)$

12) $(5-4i)+(8-4i)$

13) $(3-3i)+(-7-8i)$

14) $(i)-(2+3i)-6$

15) $(3i)(-8i)$

16) $(16i)(-2i)$

17) $(6i)(-9i)$

18) $(-7i)^2$

19) $(-5i)(-10i)$

20) $(-7-4i)(-8+6i)$

21) $(6+5i)^2$

22) $(8-6i)(-4+2i)$

23) $(-4+5i)(2-7i)$

24) $(-2+i)(3-5i)$

25) $(1+5i)(2+i)$

26) $\frac{9}{i}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 3.8 (continued)

Perform the indicated operation, writing the answer in the form of a complex number $a+bi$.

27) $\frac{5}{6i}$
28) $\frac{-3+2i}{-3i}$
29) $\frac{-3-6i}{4i}$
30) $\frac{-4+2i}{3i}$
31) $\frac{10-i}{-i}$
32) $\frac{4i}{-10+i}$
33) $\frac{8}{7-6i}$
34) $\frac{9i}{1-5i}$
35) $\frac{7}{10-7i}$
36) $\frac{4}{4+6i}$
37) $\frac{5-3i}{3+2i}$
38) $\frac{1+7i}{1+i}$
39) $\frac{6-i}{4-3i}$
40) $\frac{3+8i}{2-5i}$

ANSWERS to Practice Exercises
Section 3.8: Complex Numbers

1) $9i$

2) $3i\sqrt{5}$

---

3) -6

4) $-2\sqrt{5}$

5) $-2\sqrt{6}$

6) $9i$

---

7) $11-4i$

8) $-5-12i$

9) $-3+9i$

10) $-3-6i$

11) $-3-13i$

12) $13-8i$

13) $-4-11i$

14) $-8-2i$

15) 24

16) 32

17) 54

18) -49

19) -50

20) $80-10i$

21) $11+60i$

22) $-20+40i$

23) $27+38i$

24) $-1+13i$

25) $-3+11i$

26) $-9i$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 3.8 (continued)

27) $-\frac{5}{6}i$

28) $-\frac{2}{3}-i$

29) $-\frac{3}{2}+\frac{3}{4}i$

30) $\frac{2}{3}+\frac{4}{3}i$

31) $1+10i$

32) $\frac{4}{101}-\frac{40}{101}i$

33) $\frac{56}{85}+\frac{48}{85}i$

34) $-\frac{45}{26}+\frac{9}{26}i$

35) $\frac{70}{149}+\frac{49}{149}i$

36) $\frac{4}{13}-\frac{6}{13}i$

37) $\frac{9}{13}-\frac{19}{13}i$

38) $4+3i$

39) $\frac{27}{25}+\frac{14}{25}i$

40) $-\frac{34}{29}+\frac{31}{29}i$

Review: Chapter 3

Simplify. Assume that all variables represent positive real numbers.

1) $\sqrt{49}$
2) $\sqrt{48}$
3) $3\sqrt{45}$
4) $\sqrt{80p}$
5) $\sqrt{108x^2}$
6) $-2\sqrt{36y^4}$
7) $\sqrt{32m^2n^2}$
8) $\sqrt{147x^4y^4}$
9) $8\sqrt{243a^2b^3}$
10) $-3\sqrt{125q^4r^2s^4}$
11) $-7\sqrt{28xyz^2}$

Simplify.

12) $\sqrt[4]{-81}$
13) $\sqrt[3]{128}$
14) $-3\sqrt[4]{32}$
15) $\sqrt[6]{192x^6}$
16) $\sqrt[3]{128x^3y^6}$
17) $-4\sqrt[3]{135st^3}$

Perform the indicated operation.

18) $3\sqrt{2}+2\sqrt{2}+7\sqrt{2}$
19) $4\sqrt{7}-3\sqrt{3}+2\sqrt{3}$
20) $5\sqrt{2}+\sqrt{18}-2\sqrt{32}$
21) $-2\sqrt{32}-3\sqrt{8}-3\sqrt{12}+6\sqrt{12}$
22) $-6\sqrt[3]{54}+3\sqrt[3]{54}+11\sqrt[3]{2}$
23) $-\sqrt[4]{64}+3\sqrt[4]{4}-5\sqrt[4]{64}$

Perform the indicated operation.

24) $7\sqrt{8}\cdot\sqrt{12}$
25) $\sqrt{14}(\sqrt{7}+\sqrt{2})$
26) $(-4+\sqrt{5})(-3+7\sqrt{5})$
27) $(4+\sqrt{2})^2$
28) $\frac{\sqrt{63}}{\sqrt{7}}$

Rationalize the denominator.

29) $\frac{\sqrt{14}}{\sqrt{6}}$

30) $\frac{4\sqrt{5}}{\sqrt{15}}$

31) $\frac{-8}{\sqrt{11}+\sqrt{3}}$

32) $\frac{2\sqrt{6}}{\sqrt{2}-7}$

33) $\frac{6x^3}{7\sqrt{5x^3y^2}}$

34) $\frac{\sqrt{18n^2}}{\sqrt{12n}}$

35) $\frac{-\sqrt{5}-5\sqrt{2}}{\sqrt{2}}$

Write the expression in radical form.

36) $(13r)^{\frac{3}{2}}$

Write the expression in exponential form.

37) $\sqrt{ps}$

Evaluate.

38) $27^{-\frac{2}{3}}$

39) $121^{\frac{3}{2}}$

40) $16^{\frac{3}{4}}$

Simplify. Your answer should contain only positive exponents.

41) $x^{\frac{1}{2}}y^2 \cdot xy^{\frac{3}{2}}$

42) $\left(x^4y^{-3}\right)^0$

43) $\frac{a^{\frac{2}{5}}b^{-2} \cdot b^{\frac{5}{2}}}{5b^{-3}}$

44) $\frac{3b^{-\frac{3}{4}}(ab^{\frac{2}{3}})}{6a^{-2}b^{\frac{5}{3}}}$

45) $\frac{\left(m^2n^{\frac{1}{3}}p\right)^0}{n^{\frac{1}{3}}}$

46) $\frac{\left(x^{\frac{1}{2}}y^0\right)^{-4}}{y^3 \cdot x^{-2}}$

Write the expression using a single radical.

47) $\frac{\sqrt[3]{x^2}}{\sqrt[4]{x}}$

Solve.

48) $\sqrt{2x-3}-5=0$

49) $\sqrt{6x-8}-x=0$

50) $4+x=\sqrt{8x+25}$

51) $\sqrt{4+7x}+1=2x$

52) $\sqrt{3x-2}-\sqrt{x+5}=0$

53) $\sqrt[4]{x+2}=1$

54) $\sqrt[3]{5x-4}=-3$

Write in terms of i.

55) $\sqrt{-48}$

Multiply.

56) $\sqrt{-9}\cdot\sqrt{-25}$

57) $\sqrt{-18}\cdot\sqrt{-7}$

Perform the indicated operation, writing the answer in the form $a+bi$.

58) $(-5i)+(3i)+(2-9i)$

59) $-4i-(5-6i)$

60) $-(1+2i)-7+i$

61) $(4i)(-7i)$

62) $(-9+i)(-6-2i)$

63) $(2+3i)(4-5i)$

64) $\dfrac{11}{i}$

65) $\dfrac{3+6i}{2i}$

66) $\dfrac{3i}{-8+2i}$

67) $\dfrac{10}{7-10i}$

68) $\dfrac{2-9i}{3+i}$

ANSWERS to Review: Chapter 3

1) 7
2) $4\sqrt{3}$
3) $9\sqrt{5}$
4) $4\sqrt{5p}$
5) $6x\sqrt{3}$
6) $-12y^2$
7) $4mn\sqrt{2}$
8) $7x^2y^2\sqrt{3}$
9) $72ab\sqrt{3b}$
10) $-15q^2rs^2\sqrt{5}$
11) $-14z\sqrt{7xy}$

---

12) Not a real number
13) $4\sqrt[3]{2}$
14) $-6\sqrt[4]{2}$
15) $2x\sqrt[6]{3}$
16) $4xy^2\sqrt[3]{2}$
17) $-12t\sqrt[3]{5s}$

---

18) $12\sqrt{2}$
19) $4\sqrt{7}-\sqrt{3}$
20) 0
21) $-14\sqrt{2}+6\sqrt{3}$
22) $2\sqrt[3]{2}$
23) $-9\sqrt[4]{4}$

---

24) $28\sqrt{6}$
25) $7\sqrt{2}+2\sqrt{7}$
26) $47-31\sqrt{5}$
27) $18+8\sqrt{2}$
28) 3

29) $\frac{\sqrt{21}}{3}$

30) $\frac{4\sqrt{3}}{3}$

31) $\sqrt{3}-\sqrt{11}$

32) $\frac{-4\sqrt{3}-14\sqrt{6}}{47}$

33) $\frac{6x\sqrt{5x}}{35y}$

34) $\frac{\sqrt{6n}}{2}$

35) $\frac{-10-\sqrt{10}}{2}$

---

36) $\left(\sqrt{13r}\right)^3$

---

37) $(ps)^{\frac{1}{2}}$

---

38) $\frac{1}{9}$

39) 1331

40) 8

---

41) $x^{\frac{3}{2}}y^{\frac{7}{2}}$

42) 1

43) $\frac{a^{\frac{2}{5}}b^{\frac{7}{2}}}{5}$

44) $\frac{a^3}{2b^{\frac{7}{4}}}$

45) $\frac{1}{n^{\frac{1}{3}}}$

46) $\frac{1}{y^3}$

---

47) $\sqrt[12]{x^5}$

48) 14

49) $2, 4$

50) $-3, 3$

51) 3

52) $\frac{7}{2}$

53) -1

54) $-\frac{23}{5}$

---

55) $4i\sqrt{3}$

---

56) -15

57) $-3\sqrt{14}$

---

58) $2-11i$

59) $-5+2i$

60) $-8-i$

61) 28

62) $56+12i$

63) $23+2i$

64) $-11i$

65) $3-\frac{3}{2}i$

66) $\frac{3}{34}-\frac{6}{17}i$

67) $\frac{70}{149}+\frac{100}{149}i$

68) $-\frac{3}{10}-\frac{29}{10}i$