# CHAPTER 4

# Quadratic Equations and Graphs

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# Objectives
## Chapter 4

- Solve quadratic equations by applying the square root property.
- Solve quadratic equations by completing the square**.**
- Solve quadratic equations by using the quadratic formula.
- Graph parabolas using the vertex, $x$-intercepts, and $y$-intercept.
- Solve quadratic application problems.

# Section 4.1: Square Root Property

## Objective: Solve quadratic equations by applying the square root property.

In an earlier chapter, we learned how to solve equations by factoring. The next example reviews how we solved a quadratic equation $ax^2 + bx + c = 0$ by factoring.

**Example 1**. Solve the equation.

$$x^2 + 5x + 6 = 0 \qquad \text{Factor using } \textit{ac method}$$
$$(x+3)(x+2) = 0 \qquad \text{Set each factor equal to zero}$$
$$\begin{array}{rcr} x+3=0 & \text{or} & x+2=0 \\ \underline{-3=-3} & & \underline{-2=-2} \\ x=-3 & \text{or} & x=-2 \end{array} \qquad \text{Our Solutions}$$

However, not every quadratic equation can be solved by factoring. For example, consider the equation $x^2 - 2x - 7 = 0$. The trinomial on the left side, $x^2 - 2x - 7$, cannot be factored; however, we will see in a later section that the equation $x^2 - 2x - 7 = 0$ has two solutions: $1+2\sqrt{2}$ and $1-2\sqrt{2}$.

## SQUARE ROOT PROPERTY

In this chapter, we will learn additional methods besides factoring for solving quadratic equations. We will start with a method that makes use of the following property:

**SQUARE ROOT PROPERTY:**

If $k$ is a real number and $x^2 = k$,
then $x = \sqrt{k}$ or $x = -\sqrt{k}$

Often this property is written using shorthand notation:

If $x^2 = k$, then $x = \pm\sqrt{k}$.

To solve a quadratic equation by applying the square root property, we will first need to isolate the squared expression on one side of the equation and the constant term on the other side.

## SOLVING QUADRATIC EQUATIONS BY APPLYING THE SQUARE ROOT PROPERTY

**Example 2**. Solve the equation.

| | |
|---|---|
| $x^2 = 16$ | The squared term is already isolated; Apply the square root property $(\pm)$ |
| $\sqrt{x^2} = \pm\sqrt{16}$ | Simplify radicals |
| $x = \pm 4$ | Our Solutions |

**Example 3**. Solve the equation.

| | |
|---|---|
| $x^2 - 7 = 0$ | Isolate the squared term |
| $x^2 = 7$ | Apply the square root property $(\pm)$ |
| $\sqrt{x^2} = \pm\sqrt{7}$ | Simplify radicals |
| $x = \pm\sqrt{7}$ | Our Solutions |

**Example 4**. Solve the equation.

| | |
|---|---|
| $2x^2 + 36 = 0$ | Isolate the squared term: Subtract 36 from both sides Divide by 2 |
| $\dfrac{2x^2}{2} = \dfrac{-36}{2}$ | |
| $x^2 = -18$ | Apply the square root property $(\pm)$ |
| $\sqrt{x^2} = \pm\sqrt{-18}$ | Simplify radicals: $\pm\sqrt{-18} = \pm\sqrt{9}\cdot\sqrt{-1}\cdot\sqrt{2} = \pm 3i\sqrt{2}$ |
| $x = \pm 3i\sqrt{2}$ | Our Solutions |

**Example 5**. Solve the equation.

| | | | |
|---|---|---|---|
| | | $(2x+4)^2 = 36$ | A squared expression is already isolated on the left side; Apply the square root property $(\pm)$ |
| | | $\sqrt{(2x+4)^2} = \pm\sqrt{36}$ | Simplify radicals |
| | | $2x+4 = \pm 6$ | To avoid sign errors, separate into two equations |
| $2x+4 = 6$ | or | $2x+4 = -6$ | with one equation for $+$, one equation for $-$ |
| $-4 = -4$ | | $-4 = -4$ | Subtract $4$ from both sides |
| $\dfrac{2x}{2} = \dfrac{2}{2}$ | or | $\dfrac{2x}{2} = \dfrac{-10}{2}$ | Divide both sides by $2$ |
| $x = 1$ | or | $x = -5$ | Our Solutions |

In the previous example we used two separate equations to simplify, because when we took the root, our solutions were two rational numbers, $6$ and $-6$. If the roots do not simplify to rational numbers, we may keep the $\pm$ in the equation.

**Example 6**. Solve the equation.

| | |
|---|---|
| $(6x-9)^2 = 45$ | A squared expression is already isolated on the left side; Apply the square root property $(\pm)$ |
| $\sqrt{(6x-9)^2} = \pm\sqrt{45}$ | Simplify radicals: $\pm\sqrt{45} = \pm\sqrt{9}\cdot\sqrt{5} = \pm\, 3\sqrt{5}$ |
| $6x-9 = \pm 3\sqrt{5}$ <br> $\underline{\quad +9 = +9}$ | Use one equation because radical did not simplify to a rational number |
| $\dfrac{6x}{6} = \dfrac{9\pm 3\sqrt{5}}{6}$ | Divide both sides by $6$ |
| $x = \dfrac{9\pm 3\sqrt{5}}{6}$ | Factor numerator and denominator |
| $x = \dfrac{3\left(3\pm\sqrt{5}\right)}{3\cdot 2}$ | Divide out common factor of 3 |
| $x = \dfrac{3\pm\sqrt{5}}{2}$ | Our Solutions |

**Example 7**. Solve the equation.

| | |
|---|---|
| $2(3x-1)^2 + 7 = 23$ <br> $\underline{-7 \quad = -7}$ | Isolate the squared expression on the left side; Subtract 7 from both sides |
| $2\,(3x-1)^2 = 16$ <br> $\dfrac{2\,(3x-1)^2}{2} = \dfrac{16}{2}$ | Divide both sides by 2 |
| $(3x-1)^2 = 8$ | Apply the square root property $(\pm)$ |
| $\sqrt{(3x-1)^2} = \pm\sqrt{8}$ | Simplify radicals: $\pm\sqrt{8} = \pm\sqrt{4}\cdot\sqrt{2} = \pm\, 2\sqrt{2}$ |
| $3x-1 = \pm 2\sqrt{2}$ | Use one equation because radical did not simplify to a rational |
| $3x-1 = \pm 2\sqrt{2}$ <br> $\underline{\quad +1 = +1}$ <br> $\dfrac{3x}{3} = \dfrac{1\pm 2\sqrt{2}}{3}$ | Add $1$ to both sides <br> Divide both sides by 3 |
| $x = \dfrac{1\pm 2\sqrt{2}}{3}$ | Our Solutions |

**Example 8**. Solve the equation.

| | |
|---|---|
| $(x+3)^2+9=7$ <br> $\underline{\quad -9 \;=-9}$ | Isolate the squared expression on the left side; Subtract 9 from both sides |
| $(x+3)^2=-2$ | Apply the square root property ($\pm$) |
| $\sqrt{(x+3)^2}=\pm\sqrt{-2}$ | Simplify radicals: $\pm\sqrt{-2}=\pm\sqrt{-1}\cdot\sqrt{2}=\pm i\sqrt{2}$ |
| $x+3=\pm i\sqrt{2}$ | Use one equation because radical did not simplify to a rational |
| $x+3=\pm i\sqrt{2}$ <br> $\underline{\;-3=-3\;}$ | Subtract 3 from both sides |
| $x=-3\pm i\sqrt{2}$ | Our Solutions |

**Example 9**. Solve the equation.

| | |
|---|---|
| $\left(x+\frac{1}{3}\right)^2=\frac{2}{9}$ | Apply the square root property ($\pm$) |
| $\sqrt{\left(x+\frac{1}{3}\right)^2}=\pm\sqrt{\frac{2}{9}}$ | Simplify radicals: $\pm\sqrt{\frac{2}{9}}=\pm\frac{\sqrt{2}}{\sqrt{9}}=\pm\frac{\sqrt{2}}{3}$ |
| $x+\frac{1}{3}=\pm\frac{\sqrt{2}}{3}$ | Use one equation because radical did not simplify to a rational |
| $x+\frac{1}{3}=\pm\frac{\sqrt{2}}{3}$ | Subtract $\frac{1}{3}$ from both sides |
| $x=-\frac{1}{3}\pm\frac{\sqrt{2}}{3}$ | Add fractions |
| $x=\frac{-1\pm\sqrt{2}}{3}$ | Our Solutions |

# Practice Exercises
## Section 4.1: Square Root Property

**Solve each equation using the square root property.**

1) $x^2 = 64$
2) $x^2 = 75$
3) $x^2 + 5 = 13$
4) $x^2 - 7 = 20$
5) $x^2 + 50 = 0$
6) $5x^2 - 7 = 18$
7) $(x-4)^2 = 9$
8) $(2x+1)^2 = 25$
9) $(x+1)^2 = 3$
10) $(x-3)^2 = 12$
11) $(x+2)^2 = -9$
12) $(2x+1)^2 + 3 = 21$
13) $(9x-3)^2 = 72$
14) $(2x-8)^2 - 5 = 15$
15) $-2(x-6)^2 - 13 = 7$
16) $-3(4x-5)^2 + 8 = -19$
17) $\left(x - \frac{5}{2}\right)^2 = \frac{81}{4}$
18) $\left(x + \frac{3}{4}\right)^2 = \frac{10}{16}$

# ANSWERS to Practice Exercises
## Section 4.1: Square Root Property

1) $\pm 8$
2) $\pm 5\sqrt{3}$
3) $\pm 2\sqrt{2}$
4) $\pm 3\sqrt{3}$
5) $\pm 5i\sqrt{2}$
6) $\pm\sqrt{5}$
7) $1, 7$
8) $-3, 2$
9) $-1 \pm \sqrt{3}$
10) $3 \pm 2\sqrt{3}$
11) $-2 \pm 3i$
12) $\dfrac{-1 \pm 3\sqrt{2}}{2}$
13) $\dfrac{1 \pm 2\sqrt{2}}{3}$
14) $4 \pm \sqrt{5}$
15) $6 \pm i\sqrt{10}$
16) $\dfrac{1}{2}, 2$
17) $7, -2$
18) $\dfrac{-3 \pm \sqrt{10}}{4}$

# Section 4.2: Completing the Square

## Objective: Solve quadratic equations by completing the square.

In this section, we continue to address the question of how to solve any quadratic equation $ax^2+bx+c=0$. Now, we will learn a method known as **completing the square**. When completing the square, we will change the quadratic into a perfect square that can then be solved by applying the square root property. The next example reviews the square root property.

**Example 1**. Solve the equation.

| | |
|---|---|
| $(x+5)^2=18$ | Use square root property |
| $\sqrt{(x+5)^2}=\pm\sqrt{18}$ | Simplify radicals |
| $x+5=\pm3\sqrt{2}$ | Subtract 5 from both sides |
| $-5=-5$ | |
| $x=-5\pm3\sqrt{2}$ | Our Solutions |

## COMPLETING THE SQUARE

To complete the square, or make our problem into the form of the previous example, we will be searching for the third term of a trinomial. If a quadratic is of the form $x^2+bx+c$, and a perfect square, the third term, $c$, can be easily found by the formula $(\frac{1}{2}\cdot b)^2$. This is shown in the following examples, where we find the number that completes the square, and then factor that perfect square trinomial.

**Example 2**. Find the value of $c$ that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

| | |
|---|---|
| $x^2+8x+c$ | $c=\left(\frac{1}{2}\cdot b\right)^2$ and our $b=8$ |
| | The third term that completes the square is 16: $\left(\frac{1}{2}\cdot 8\right)^2=(4)^2=16$ |
| $x^2+8x+16$ $=(x+4)(x+4)$ | Our expression is a perfect square; factor |
| $=(x+4)^2$ | Our Answer |

**Example 3**. Find the value of $c$ that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

$x^2 - 7x + c$ $\quad$ $c = \left(\frac{1}{2}\cdot b\right)^2$ and our $b = -7$

The third term that completes the square is $\frac{49}{4}$ :

$$\left(\frac{1}{2}\cdot -7\right)^2 = \left(-\frac{7}{2}\right)^2 = \frac{49}{4}$$

$x^2 - 7x + \frac{49}{4}$ $\quad$ Our expression is a perfect square; factor

$= \left(x - \frac{7}{2}\right)\left(x - \frac{7}{2}\right)$

$= \left(x - \frac{7}{2}\right)^2$ $\quad$ Our Answer

**Example 4**. Find the value of $c$ that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

$x^2 + \frac{5}{3}x + c$ $\quad$ $c = \left(\frac{1}{2}\cdot b\right)^2$ and our $b = \frac{5}{3}$

The third term that completes the square is $\frac{25}{36}$ :

$$\left(\frac{1}{2}\cdot\frac{5}{3}\right)^2 = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

$x^2 + \frac{5}{3}x + \frac{25}{36}$ $\quad$ Our expression is a perfect square; factor

$= \left(x + \frac{5}{6}\right)^2$ $\quad$ Our Answer

## SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

The process in the previous examples, combined with the square root property, is used to solve quadratic equations by completing the square. The following six steps describe the process used to solve a quadratic equation by completing the square, along with a practice example to demonstrate each step.

| Steps for Solving by Completing the Square | Example |
|---|---|
| | $3x^2+18x-6=0$ |
| 1. Separate the constant term from the variable terms. | $\begin{aligned} +6&=+6 \\ \hline 3x^2+18x \quad &=6 \end{aligned}$ |
| 2. If $a \neq 1$, then divide each term by $a$. | $\frac{3}{3}x^2+\frac{18}{3}x = \frac{6}{3}$ <br> $x^2+6x = 2$ |
| 3. Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$. | $\left(\frac{1}{2}\cdot 6\right)^2 = 3^2 = 9$ |
| 4. Add the resulting value to both sides of the equation. | $x^2+6x = 2$ <br> $+9=+9$ <br> $x^2+6x+9=11$ |
| 5. Factor the perfect square trinomial. | $(x+3)^2=11$ |
| 6. Solve by applying the square root property. | $\sqrt{(x+3)^2}=\pm\sqrt{11}$ <br> $x+3=\pm\sqrt{11}$ <br> $-3=-3$ <br> $x=-3\pm\sqrt{11}$ |

The advantage of this method is that it can be used to solve **any** quadratic equation. The following examples show how completing the square can give us rational solutions, irrational solutions, and even complex solutions.

**Example 5**. Solve the equation by completing the square.

$$2x^2+20x+48=0$$
$$\underline{-48=-48}$$

Separate the constant term from variable terms
Subtract 48 from both sides of the equation

$$\frac{2x^2}{2}+\frac{20x}{2} = \frac{-48}{2}$$

Divide each term by 2

$$x^2+10x = -24$$

Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$

Our $b=10$; $\left(\frac{1}{2}\cdot 10\right)^2 = (5)^2 = 25$

$$x^2+10x = -24$$
$$\underline{+25=+25}$$

Add 25 to both sides of the equation

$$x^2+10x+25=1$$

Factor the perfect square trinomial

$$(x+5)^2=1$$

Solve by applying the square root property

| | |
|---|---|
| $\sqrt{(x+5)^2} = \pm\sqrt{1}$ | Simplify radicals |
| $x+5=\pm 1$ | One equation for $+$, one equation for $-$ |
| $x+5=1$ or $x+5=-1$ | Subtract $5$ from both sides |
| $-5=-5$ $\quad$ $-5=-5$ | |
| $x=-4$ or $x=-6$ | Our Solutions |

**Example 6**. Solve the equation by completing the square.

| | |
|---|---|
| $x^2-3x-2=0$ | Separate the constant term from variable terms |
| $+2 = +2$ | Add 2 to both sides |
| $x^2-3x = 2$ | No need to divide since $a=1$ |
| | Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$ |
| | Our $b=-3$; $\left(\frac{1}{2}\cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ |
| | Add $\frac{9}{4}$ to both sides of the equation |
| $x^2-3x+\frac{9}{4}=2+\frac{9}{4}$ | Need common denominator (4) on right side of equation |
| | $\frac{2}{1}\left(\frac{4}{4}\right)+\frac{9}{4}=\frac{8}{4}+\frac{9}{4}=\frac{17}{4}$ |
| $x^2-3x+\frac{9}{4}=\frac{17}{4}$ | Factor the perfect square trinomial |
| $\left(x-\frac{3}{2}\right)^2=\frac{17}{4}$ | Solve by applying the square root property |
| $\sqrt{\left(x-\frac{3}{2}\right)^2}=\pm\sqrt{\frac{17}{4}}$ | Simplify radicals |
| $x-\frac{3}{2}=\pm\frac{\sqrt{17}}{2}$ | Add $\frac{3}{2}$ to both sides of the equation |
| $+\frac{3}{2} = +\frac{3}{2}$ | Notice that we already have a common denominator |
| $x=\frac{3\pm\sqrt{17}}{2}$ | Our Solutions |

As Example 6 has shown, when solving by completing the square, we will often need to use fractions and be comfortable finding common denominators and adding fractions together.

Sometimes when solving quadratic equations, the solutions are complex numbers, as is the case in Example 7.

**Example 7**. Solve the equation by completing the square.

| | |
|---|---|
| $x^2 - 6x + 30 = 0$ | Separate the constant term from variable terms |
| $\underline{\quad -30 = -30}$ | Subtract $30$ from both sides of the equation |
| $x^2 - 6x \quad = -30$ | No need to divide since $a = 1$ |
| | Find the value that completes the square: $\left(\frac{1}{2} \cdot b\right)^2$ |
| | Our $b = -6$; $\left(\frac{1}{2} \cdot -6\right)^2 = (-3)^2 = 9$ |
| $x^2 - 6x \quad = -30$ | Add $9$ to both sides of the equation |
| $\underline{\quad +9 = +9}$ | |
| $x^2 - 6x + 9 = -21$ | Factor the perfect square trinomial |
| $(x-3)^2 = -21$ | Solve by applying the square root property |
| $\sqrt{(x-3)^2} = \pm\sqrt{-21}$ | Simplify radicals |
| $x - 3 = \pm i\sqrt{21}$ | Add $3$ to both sides |
| $\underline{\quad +3 = +3}$ | |
| $x = 3 \pm i\sqrt{21}$ | Our Solutions |

We can use completing the square to solve any quadratic equation so we want to get comfortable using the six steps of this method.

# Practice Exercises
## Section 4.2: Completing the Square

**Find the value of $c$ that makes each expression a perfect square trinomial; then, factor that perfect square trinomial.**

1) $x^2 - 30x + c$
2) $a^2 + 24a + c$
3) $m^2 - 36m + c$
4) $x^2 + 34x + c$
5) $x^2 - 15x + c$
6) $r^2 + \frac{1}{9}r + c$
7) $y^2 + y + c$
8) $p^2 - 17p + c$

**Solve each equation by completing the square.**

9) $x^2 - 16x + 55 = 0$
10) $n^2 - 8n - 9 = 0$
11) $v^2 - 8v + 45 = 0$
12) $b^2 + 2b + 43 = 0$
13) $x^2 + 5x = 7$
14) $3k^2 + 2k - 4 = 0$
15) $-4z^2 + z + 1 = 0$
16) $8a^2 + 16a - 1 = 0$
17) $x^2 + 10x - 57 = 4$
18) $p^2 - 16p - 52 = 0$
19) $n^2 - 16n + 67 = 4$
20) $m^2 - 8m - 12 = 0$
21) $2x^2 + 4x + 38 = -6$
22) $6r^2 + 12r - 24 = -6$
23) $8b^2 + 16b - 37 = 5$
24) $6n^2 - 12n - 14 = 4$
25) $x^2 = -10x - 29$
26) $v^2 = 14v + 36$
27) $3k^2 + 9 = 6k$
28) $5n^2 = -10n + 15$
29) $p^2 - 8p = -55$
30) $x^2 + 8x + 15 = 8$
31) $7n^2 - n + 7 = 7n + 6n^2$
32) $n^2 + 4n = 12$
33) $8n^2 + 16n = 64$
34) $b^2 + 7b - 33 = 0$
35) $-5x^2 - 8x + 40 = -8$
36) $m^2 = -15 + 9m$
37) $4b^2 - 15b + 56 = 3b^2$
38) $10v^2 - 15v = 27 + 4v^2 - 6v$
39) $n^2 = -21 + 10n$
40) $a^2 - 56 = -10a$

# ANSWERS to Practice Exercises
## Section 4.2: Completing the Square

1) $225; (x-15)^2$
2) $144; (a+12)^2$
3) $324; (m-18)^2$
4) $289; (x+17)^2$
5) $\frac{225}{4}; \left(x-\frac{15}{2}\right)^2$
6) $\frac{1}{324}; \left(r+\frac{1}{18}\right)^2$
7) $\frac{1}{4}; \left(y+\frac{1}{2}\right)^2$
8) $\frac{289}{4}; \left(p-\frac{17}{2}\right)^2$

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9) $11, 5$
10) $9, -1$
11) $4+i\sqrt{29}, 4-i\sqrt{29}$
12) $-1+i\sqrt{42}, -1-i\sqrt{42}$
13) $\frac{-5+\sqrt{53}}{2}, \frac{-5-\sqrt{53}}{2}$
14) $\frac{-1+\sqrt{13}}{3}, \frac{-1-\sqrt{13}}{3}$
15) $\frac{1+\sqrt{17}}{8}, \frac{1-\sqrt{17}}{8}$
16) $\frac{-4+3\sqrt{2}}{4}, \frac{-4-3\sqrt{2}}{4}$
17) $-5+\sqrt{86}, -5-\sqrt{86}$
18) $8+2\sqrt{29}, 8-2\sqrt{29}$
19) $9, 7$
20) $4+2\sqrt{7}, 4-2\sqrt{7}$
21) $-1+i\sqrt{21}, -1-i\sqrt{21}$
22) $1, -3$
23) $\frac{3}{2}, -\frac{7}{2}$
24) $3, -1$
25) $-5+2i, -5-2i$
26) $7+\sqrt{85}, 7-\sqrt{85}$
27) $1+i\sqrt{2}, 1-i\sqrt{2}$
28) $1, -3$
29) $4+i\sqrt{39}, 4-i\sqrt{39}$
30) $-1, -7$
31) $7, 1$
32) $2, -6$
33) $2, -4$
34) $\frac{-7+\sqrt{181}}{2}, \frac{-7-\sqrt{181}}{2}$
35) $\frac{12}{5}, -4$
36) $\frac{9+\sqrt{21}}{2}, \frac{9-\sqrt{21}}{2}$
37) $8, 7$
38) $3, -\frac{3}{2}$
39) $7, 3$
40) $4, -14$

# Section 4.3: Quadratic Formula

## Objective: Solve quadratic equations using the quadratic formula.

In this section, we will develop a formula to solve any quadratic equation $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$. Solve for this general equation for $x$ by completing the square:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ -c &= -c \end{aligned}$$

Separate the constant term from variable terms

Subtract $c$ from both sides

$$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a}$$

Divide each term by $a$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

Find the value that completes the square:

$$\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Add that value to both sides of the equation

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Subtract fractions on the right side of the equation using a common denominator of $4a^2$:

$$\frac{b^2}{4a^2} - \frac{c}{a}\left(\frac{4a}{4a}\right) = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Factor the perfect square trinomial

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Solve by applying the square root property

$$\sqrt{\left(x + \frac{b}{2}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Simplify radicals

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Our Solutions

## QUADRATIC FORMULA

This result is a very important one to us because we can use this formula to solve any quadratic equation. Once we identify the values of *a; b;* and *c* in the quadratic equation, we can substitute those values into $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ and we get our solutions. This formula is known as the **quadratic formula**.

**QUADRATIC FORMULA:**

The solutions to the quadratic equation $ax^2+bx+c=0$ for $a\neq 0$ are given by the formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

## SOLVING QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA

We can use the quadratic formula to solve any quadratic equation. This method is demonstrated in the following examples.

**Example 1**. Solve the equation using the quadratic formula.

$x^2+3x+2=0$ — $a=1$, $b=3$, $c=2$; use quadratic formula

$x=\dfrac{-(3)\pm\sqrt{(3)^2-4(1)(2)}}{2(1)}$ — Evaluate the exponent and multiply

$x=\dfrac{-3\pm\sqrt{9-8}}{2}$ — Evaluate the subtraction under radical sign

$x=\dfrac{-3\pm\sqrt{1}}{2}$ — Evaluate the root

$x=\dfrac{-3\pm 1}{2}$ — Evaluate $+$ and $-$ to get the two answers

$x=\dfrac{-3+1}{2}$ or $x=\dfrac{-3-1}{2}$

$x=\dfrac{-2}{2}$ or $x=\dfrac{-4}{2}$ — Simplify the fractions, if possible

$x=-1$ or $-2$ — Our Solutions

As we are solving a quadratic equation using the quadratic formula, it is important to remember that the equation must first be set equal to zero.

**Example 2**. Solve the equation using the quadratic formula.

| | |
|---|---|
| $25x^2 = 30x + 11$ | First set the equation equal to zero |
| $-30x - 11 = -30x - 11$ | Subtract $30x$ and 11 from both sides of the equation |
| $25x^2 - 30x - 11 = 0$ | $a = 25$, $b = -30$, and $c = -11$; use quadratic formula |
| $x = \dfrac{-(-30) \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)}$ | Evaluate the exponent and multiply |
| $x = \dfrac{30 \pm \sqrt{900 + 1100}}{50}$ | Evaluate the addition under radical sign |
| $x = \dfrac{30 \pm \sqrt{2000}}{50}$ | Simplify the root |
| $x = \dfrac{30 \pm 20\sqrt{5}}{50}$ | Factor numerator and denominator |
| $x = \dfrac{10\left(3 \pm 2\sqrt{5}\right)}{10 \cdot 5}$ | Divide out common factor of 10 |
| $x = \dfrac{3 \pm 2\sqrt{5}}{5}$ | Our Solutions |

**Example 3**. Solve the equation using the quadratic formula.

| | |
|---|---|
| $3x^2 + 4x + 8 = 2x^2 + 6x - 5$ | First set the equation equal to zero |
| $-2x^2 - 6x + 5 = -2x^2 - 6x + 5$ | Subtract $2x^2$ and $6x$, and add $5$ |
| $x^2 - 2x + 13 = 0$ | $a = 1$, $b = -2$, and $c = 13$; use quadratic formula |
| $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)}$ | Evaluate the exponent and multiply |
| $x = \dfrac{2 \pm \sqrt{4 - 52}}{2}$ | Evaluate the subtraction under radical sign |

$$x = \frac{2 \pm \sqrt{-48}}{2}$$ Simplify the root

$$x = \frac{2 \pm 4i\sqrt{3}}{2}$$ Factor numerator

$$x = \frac{2\left(1 \pm 2i\sqrt{3}\right)}{2 \cdot 1}$$ Divide out common factor of 2

$$x = 1 \pm 2i\sqrt{3}$$ Our Solutions

Notice this equation has two imaginary solutions and they are complex conjugates.

When we solve quadratic equations, we don't necessarily get two unique solutions. We can end up with only one real number solution if the square root simplifies to zero.

**Example 4**. Solve the equation using the quadratic formula.

$$4x^2 - 12x + 9 = 0$$ $a = 4$, $b = -12$, $c = 9$; use quadratic formula

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$ Evaluate the exponent and multiply

$$x = \frac{12 \pm \sqrt{144 - 144}}{8}$$ Evaluate the subtraction under radical sign

$$x = \frac{12 \pm \sqrt{0}}{8}$$ Simplify the root

$$x = \frac{12 \pm 0}{8}$$ Evaluate $+$ and $-$ to get the two answers;

They are identical values; so, only one instance needs to be considered

$$x = \frac{12}{8}$$ Reduce fraction

$$x = \frac{3}{2}$$ Our Solution

When solving a quadratic equation, if the term with $x$ or the constant term is missing, we can still solve the equation using the quadratic formula. We simply use zero for the coefficient of the missing term. If the term with $x$ is missing, we have $b = 0$ and if the constant term is missing, we have $c = 0$. Note if $a = 0$, the term with $x^2$ is missing, meaning the equation is a linear equation, not a quadratic equation.

**Example 5**. Solve the equation using the quadratic formula.

| | |
|---|---|
| $3x^2 + 7 = 0$ | $a = 3$, $b = 0$ (missing term), $c = 7$; use quadratic formula |
| $x = \dfrac{-(0) \pm \sqrt{(0)^2 - 4(3)(7)}}{2(3)}$ | Evaluate the exponent and multiply |
| $x = \pm\dfrac{\sqrt{-84}}{6}$ | Simplify the root |
| $x = \pm\dfrac{2i\sqrt{21}}{6}$ | Reduce the fraction; divide $2i$ and $6$ by $2$ |
| $x = \pm\dfrac{i\sqrt{21}}{3}$ | Our Solutions |

## SELECTING A METHOD FOR SOLVING A QUADRATIC EQUATION

We have covered four different methods that can be used to solve a quadratic equation: factoring, applying the square root property, completing the square, and using the quadratic formula. It is important to be familiar with all four methods as each has its own advantages when solving quadratic equations.

Some of the examples in this section could have been solved using a method other than the quadratic formula. In Example 1, we used the quadratic formula to solve the equation $x^2 + 3x + 2 = 0$. We could have chosen to solve this equation factoring instead:

$$x^2 + 3x + 2 = 0$$
$$(x+1)(x+2) = 0$$
$$x + 1 = 0 \qquad x + 2 = 0$$
$$x = -1 \qquad x = -2$$

In Example 5, we could have chosen to solve $3x^2 + 7 = 0$ by applying the square root property since there is no $x$ term and we can isolate the squared term.

The following table walks you through a suggested process and an example of each method to decide which would be best to use when solving a quadratic equation.

| | |
|---|---|
| 1. If $ax^2+bx+c$ can be factored easily, solve by **factoring**: | $x^2-5x+6=0$<br>$(x-2)(x-3)=0$<br>$x=2$ or $x=3$ |
| 2. If the equation can be written with a squared term or expression on one side and a constant term on the other, solve by applying the **square root property**: | $x^2-7=0$<br>$x^2=7$<br>$x=\pm\sqrt{7}$ |
| 3. If $a=1$ and $b$ is even, solve by **completing the square**: | $x^2+2x=4$<br>$\left(\frac{1}{2}\cdot 2\right)^2=1^2=1$<br>$x^2+2x+1=4+1$<br>$(x+1)^2=5$<br>$x+1=\pm\sqrt{5}$<br>$x=-1\pm\sqrt{5}$ |
| 4. Otherwise, solve by the **quadratic formula**: | $x^2-3x+4=0$<br>$x=\dfrac{3\pm\sqrt{(-3)^2-4(1)(4)}}{2(1)}$<br>$x=\dfrac{3\pm i\sqrt{7}}{2}$ |

The above table offers a suggestion for deciding how to solve a quadratic equation. Remember that the methods of completing the square and the quadratic formula will always work to solve any quadratic equation. Solving a quadratic equation by factoring only works if the expression can be factored.

# Practice Exercises
## Section 4.3: Quadratic Formula

**Solve each equation using the quadratic formula.**

1) $x^2-4x+3=0$
2) $m^2+4m-48=-3$
3) $4x^2-7=-5x$
4) $3k^2=-3k+11$
5) $2x^2+4x=-4$
6) $5p^2+2p+6=0$
7) $3r^2-2r-1=0$
8) $2x^2-2x-15=0$
9) $4n^2-36=0$
10) $3b^2+6=0$
11) $2x^2+3x+8=0$
12) $3x=6x^2+7$
13) $2x^2=8x+2$
14) $-16t^2+32t+48=0$
15) $7x^2+3x=14$
16) $6n^2-1=0$
17) $2p^2+6p-16=4$
18) $9m^2-16=0$
19) $3n^2+3n=-3$
20) $3t^2-3=8t$
21) $2x^2=-7x+49$
22) $-3r^2+4=-6r$
23) $5x^2=7x+7$
24) $6a^2=-5a+13$
25) $8n^2=-3n-8$
26) $6v^2=4+6v$
27) $2x^2+5x=-3$
28) $x^2=8$
29) $4a^2-64=0$
30) $2k^2+6k-16=2k$
31) $4p^2+5p-36=3p^2$
32) $12x^2+x+7=5x^2+5x$
33) $-5n^2-3n-52=2-7n^2$
34) $7m^2-6m+6=-m$
35) $7r^2-12=-3r$
36) $3x^2-3=x^2$
37) $2n^2-9=4$
38) $6t^2=t^2+7-t$

# ANSWERS to Practice Exercises

## Section 4.3: Quadratic Formula

1) $1, 3$
2) $5, -9$
3) $\frac{-5+\sqrt{137}}{8}, \frac{-5-\sqrt{137}}{8}$
4) $\frac{-3+\sqrt{141}}{6}, \frac{-3-\sqrt{141}}{6}$
5) $-1+i, -1-i$
6) $\frac{-1+i\sqrt{29}}{5}, \frac{-1-i\sqrt{29}}{5}$
7) $1, -\frac{1}{3}$
8) $\frac{1+\sqrt{31}}{2}, \frac{1-\sqrt{31}}{2}$
9) $3, -3$
10) $i\sqrt{2}, -i\sqrt{2}$
11) $\frac{-3+i\sqrt{55}}{4}, \frac{-3-i\sqrt{55}}{4}$
12) $\frac{3+i\sqrt{159}}{12}, \frac{3-i\sqrt{159}}{12}$
13) $2+\sqrt{5}, 2-\sqrt{5}$
14) $3, -1$
15) $\frac{-3+\sqrt{401}}{14}, \frac{-3-\sqrt{401}}{14}$
16) $\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}$
17) $2, -5$
18) $\frac{4}{3}, -\frac{4}{3}$
19) $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$
20) $3, -\frac{1}{3}$
21) $\frac{7}{2}, -7$
22) $\frac{3+\sqrt{21}}{3}, \frac{3-\sqrt{21}}{3}$
23) $\frac{7+3\sqrt{21}}{10}, \frac{7-3\sqrt{21}}{10}$
24) $\frac{-5+\sqrt{337}}{12}, \frac{-5-\sqrt{337}}{12}$
25) $\frac{-3+i\sqrt{247}}{16}, \frac{-3-i\sqrt{247}}{16}$
26) $\frac{3+\sqrt{33}}{6}, \frac{3-\sqrt{33}}{6}$
27) $-1, -\frac{3}{2}$
28) $2\sqrt{2}, -2\sqrt{2}$
29) $4, -4$
30) $2, -4$
31) $4, -9$
32) $\frac{2+3i\sqrt{5}}{7}, \frac{2-3i\sqrt{5}}{7}$
33) $6, -\frac{9}{2}$
34) $\frac{5+i\sqrt{143}}{14}, \frac{5-i\sqrt{143}}{14}$
35) $\frac{-3+\sqrt{345}}{14}, \frac{-3-\sqrt{345}}{14}$
36) $\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}$
37) $\frac{\sqrt{26}}{2}, -\frac{\sqrt{26}}{2}$
38) $\frac{-1+\sqrt{141}}{10}, \frac{-1-\sqrt{141}}{10}$

# Section 4.4: Parabolas

**Objective: Graph parabolas using the vertex, *x*-intercepts, and *y*-intercept.**

Just as the graph of a linear equation $y = mx + b$ can be drawn, the graph of a quadratic equation $y = ax^2 + bx + c$ can be drawn. The graph is simply a picture showing what pairs of values $x$ and $y$ can be used to make the equation true. For a linear equation, the graph is a line but for a quadratic equation, the graph is a U shaped curve called a **parabola**.

## GRAPHING A PARABOLA BY CREATING A TABLE OF VALUES

One way to draw the graph of a quadratic equation is to make a table of values and evaluate the equation for each *x*-value we choose. The completed table gives us a set of points to graph. Remember that points are ordered pairs in the form of $(x, y)$; so, each *x*-value and its corresponding *y*-value are a point to be graphed.

**Example 1**. Graph the parabola $y = x^2 - 4x + 3$.

Make a table of values. We will test five *x*-values to get an idea of the shape of the graph:

$$y = x^2 - 4x + 3$$

| $x$ | $y$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

Plug $0$ in for $x$ and evaluate: $y = (0)^2 + 4(0) + 3 = 0 - 0 + 3 = 3$

Plug $1$ in for $x$ and evaluate: $y = (1)^2 - 4(1) + 3 = 1 - 4 + 3 = 0$

Plug $2$ in for $x$ and evaluate: $y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$

Plug $3$ in for $x$ and evaluate: $y = (3)^2 - 4(3) + 3 = 9 - 12 + 3 = 0$

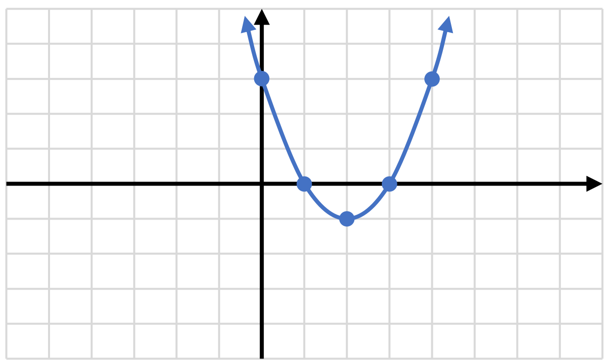
Plug $4$ in for $x$ and evaluate: $y = (4)^2 - 4(4) + 3 = 16 - 16 + 3 = 3$

The completed table is shown below:

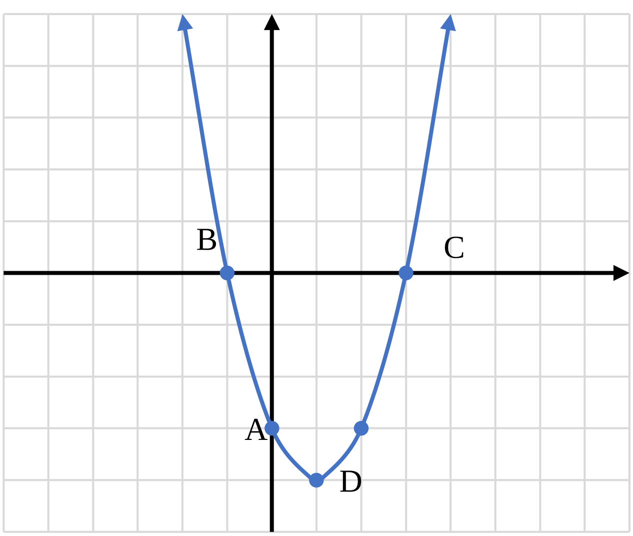
$$y = x^2 - 4x + 3$$

| $x$ | $y$ |
|---|---|
| 0 | 3 |
| 1 | 0 |
| 2 | $-1$ |
| 3 | 0 |
| 4 | 3 |

Graph by plotting the points $(0,3)$, $(1,0)$, $(2,-1)$, $(3,0)$ and $(4,3)$. Connect the points with a smooth, U shaped curve.



The above method to graph a parabola works for any quadratic equation; however, it can be very tedious to find all the points that would be necessary to get the correct bend and shape. For this reason, we identify several key points on a graph to help us graph parabolas more efficiently. These key points are described below.




$y$ **-intercept** (Point A): where the graph crosses the vertical $y$-axis.

$x$ **-intercepts** (Points B and C): where the graph crosses the horizontal $x$-axis.

**Vertex** (Point D): the turning point where the graph changes directions.

## GRAPHING A PARABOLA USING THE VERTEX, *X*-INTERCEPTS, AND *Y*-INTERCEPT

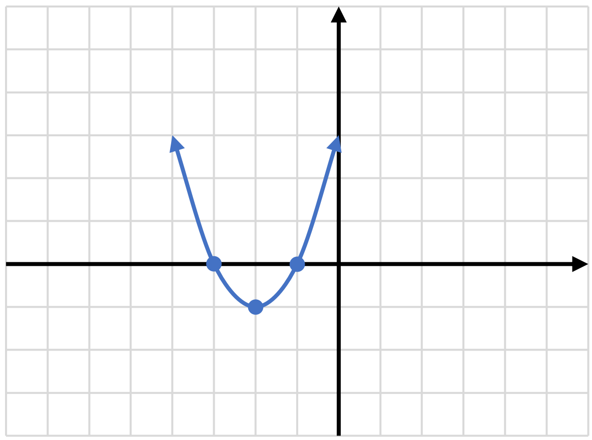
We will use the following method to find each of the points on our parabola.

**To graph the parabola $y = ax^2 + bx + c$:**

1. **Find the $y$-intercept:** Find the $y$-intercept by evaluating when $x = 0$; this always simplifies to $y = c$.
2. **Find the $x$-intercepts:** Find any $x$-intercepts by setting $y = 0$ and solving the equation $0 = ax^2 + bx + c$.
   If the solutions are real numbers, there are two $x$-intercepts. It is also possible to have only one $x$-intercept or no $x$-intercepts (if the solutions are complex numbers).
3. **Find the vertex:** Let $x = \frac{-b}{2a}$ to find the $x$-coordinate of the vertex. Then plug this $x$-value into the equation to find its corresponding *y*-value, which is the *y*-coordinate of the vertex.
4. **Determine whether the parabola opens upward or downward:**
   If $a$ is a *positive* number, then the vertex will be the *minimum* point of the parabola and the graph will open *upward* (U-shaped).
   If $a$ is a *negative* number, then the vertex will be the *maximum* point of the parabola and the graph will open *downward* (upside down U-shaped).
5. **Plot the points and connect with a smooth U-shaped curve.**

**Example 2**. Graph the parabola

| | |
|---|---|
| $y = x^2 + 4x + 3$ | Find the key points |
| $y = 3$ | *y*-intercept is $y = c$, point $(0, 3)$ |
| $0 = x^2 + 4x + 3$ | To find the $x$-intercepts, we solve the equation |
| $0 = (x+3)(x+1)$ | Factor completely |
| $x + 3 = 0$ or $x + 1 = 0$ | Set each factor equal to zero |
| $-3 = -3$    $-1 = -1$ | Solve each equation |
| $x = -3$ or $x = -1$ | Our $x$-intercepts, points $(-3, 0)$ and $(-1, 0)$ |
| $x = \frac{-4}{2(1)} = \frac{-4}{2} = -2$ | To find the vertex, first use $x = \frac{-b}{2a}$ |
| $y = (-2)^2 + 4(-2) + 3$ | Plug this value into the equation to find the $y$-coordinate |
| $y = 4 - 8 + 3$ | Evaluate |
| $y = -1$ | $y$-value of vertex |
| $(-2, -1)$ | Vertex as a point |



Graph points $(0,3)$, $(-3,0)$, and $(-1,0)$, as well as the vertex at $(-2,-1)$.

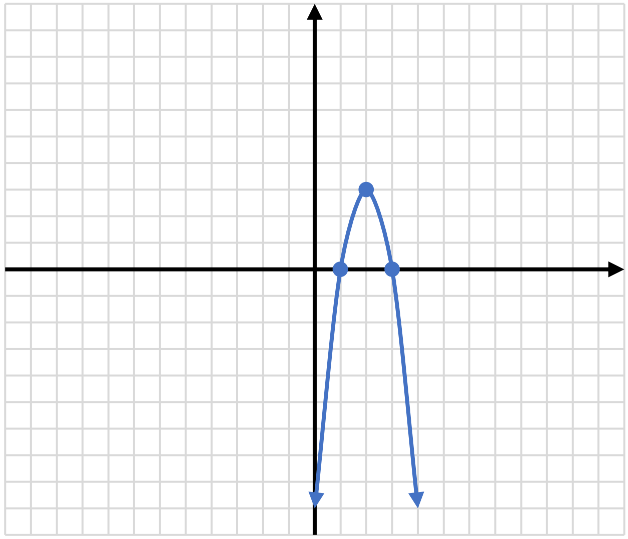
Connect the dots with a smooth curve in a U shape to get our parabola.

Our Graph

If the leading coefficient $a$ in $y = ax^2 + bx + c$ is *negative*, the parabola will end up having an upside-down U shape. The process to graph it is identical, we just need to be very careful of how our signs operate. Remember, if $a$ is negative, then $ax^2$ will also be negative because we only square the $x$, not the $a$.

**Example 3**. Graph the parabola

| | |
|---|---|
| $y = -3x^2 + 12x - 9$ | Find the key points |
| $y = -9$ | $y$-intercept is $y = c$, point $(0,-9)$ |
| $0 = -3x^2 + 12x - 9$ | To find the $x$-intercepts, this equation |
| $0 = -3(x^2 - 4x + 3)$ | Factor out GCF first, then factor rest |
| $0 = -3(x-3)(x-1)$ | Set each factor with a variable equal to zero |
| $x - 3 = 0$ or $x - 1 = 0$ | Solve each equation |
| $+3 = +3$ $\quad$ $+1 = +1$ | |
| $x = 3$ or $x = 1$ | Our $x$-intercepts, points $(3,0)$ and $(1,0)$ |
| $x = \dfrac{-12}{2(-3)} = \dfrac{-12}{-6} = 2$ | To find the vertex, first use $x = \dfrac{-b}{2a}$ |
| $y = -3(2)^2 + 12(2) - 9$ | Plug this value into the equation to find the $y$-coordinate |
| $y = -3(4) + 24 - 9$ | Evaluate |
| $y = -12 + 24 - 9$ | |
| $y = 3$ | $y$-value of vertex |
| $(2,3)$ | Vertex as a point |



Graph the points $(0,-9)$, $(3,0)$, and $(1,0)$, as well as the vertex at $(2,3)$.

Connect the dots with a smooth curve in an upside-down U shape to get our parabola.

Our Graph

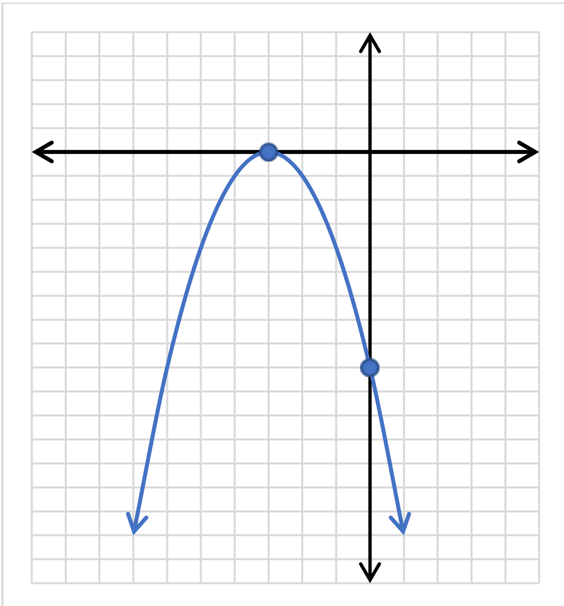
It is important to remember the graph of all quadratics is a parabola with the same U shape (either opening up or opening down). If you plot your points and they cannot be connected in the correct U shape, then at least one of your points must be wrong. Go back and check your work!

Whenever you have a perfect square trinomial quadratic equation, you will have only one unique *x*-intercept, and that *x*-intercept will also be the vertex of the parabola.

**Example 4**. Graph the parabola

| | |
|---|---|
| $y=-x^2-6x-9$ | Find the key points |
| $y=-9$ | *y*-intercept is $y=c$, point $(0,-9)$ |
| $0=-x^2-6x-9$ | To find the *x*-intercept in this equation, |
| $0=-1(x^2+6x+9)$ | Factor out GCF first, then factor the trinomial |
| $0=-1(x+3)(x+3)$ | Set each factor with a variable equal to zero |
| $x+3=0$ or $x+3=0$ | Solve each equation |
| $\underline{-3=-3}$ $\underline{-3=-3}$ | |
| $x=-3$ or $x=-3$ | Since they are the same value, the *x*-intercept is $(-3,0)$ |
| $x=-\frac{-6}{2(-1)}=-\frac{-6}{-2}=-3$ | To find the vertex, first use $x=\frac{-b}{2a}$ |
| $y=-(-3)^2-6(-3)-9$ | Plug this value into the equation to find the *y*-coordinate |
| $y=-(9)+18-9$ | Evaluate |
| $y=-9+18-9$ | |
| $y=0$ | *y*-value of vertex |
| $(-3,0)$ | Vertex as a point |

Notice that the *x*-intercept and the vertex are the same point $(-3, 0)$. This occurs whenever you have a perfect square trinomial as your quadratic equation. This is because whenever you factor a perfect square trinomial, both factors are identical. By setting each factor equal to zero there is only one unique solution.



Graph the *y*-intercept $(0, -9)$ and the vertex $(-3, 0)$.

Connect the dots with a smooth curve in an upside-down U shape to get our parabola.

Our Graph

It is important to remember the graphs of all quadratics are parabolas with the same basic U shape. The differences come from the vertex being shifted to a different location, the curve opening up or down, and how quickly the curve opens.

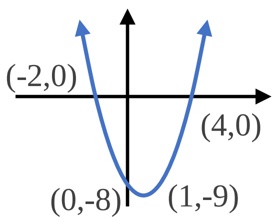
# Practice Exercises
## Section 4.4: Parabolas

**Find the vertex and intercepts. Use this information to graph each parabola.**

1) $y = x^2 - 2x - 8$
2) $y = x^2 - 2x - 3$
3) $y = 2x^2 - 12x + 10$
4) $y = 2x^2 - 12x + 16$
5) $y = -2x^2 + 12x - 18$
6) $y = -2x^2 + 12x - 10$
7) $y = -3x^2 + 24x - 45$
8) $y = -3x^2 + 12x - 9$
9) $y = -x^2 + 4x + 5$
10) $y = -x^2 + 4x - 3$
11) $y = -x^2 + 6x - 5$
12) $y = -2x^2 + 16x - 30$
13) $y = -2x^2 + 16x - 24$
14) $y = 2x^2 + 4x - 6$
15) $y = 3x^2 + 12x + 9$
16) $y = 5x^2 + 30x + 45$
17) $y = 5x^2 - 40x + 75$
18) $y = 5x^2 + 20x + 15$
19) $y = -5x^2 - 60x - 175$
20) $y = -5x^2 + 20x - 15$
21) $y = 3x^2 - 6x + 1$
22) $y = 9x^2 - 18x + 4$
23) $y = -6x^2 - 18x - 11$
24) $y = x^2 - 4x + 5$
25) $y = -3x^2 + 6x - 5$
26) $y = x^2 + 6x + 10$
27) $y = x^2 + 8x + 16$
28) $y = -x^2 + 10x - 25$

# ANSWERS to Practice Exercises
## Section 4.4: Parabolas

1)

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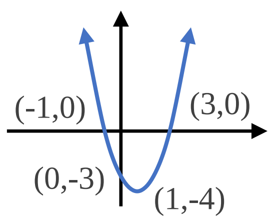


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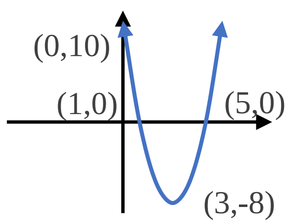


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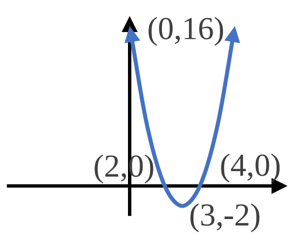
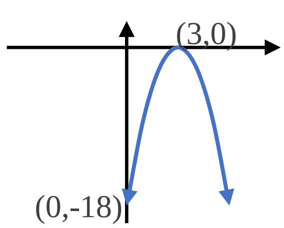
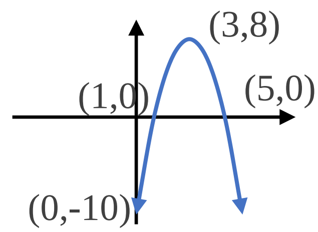


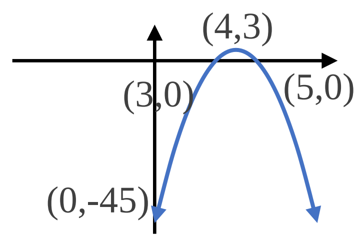

5)



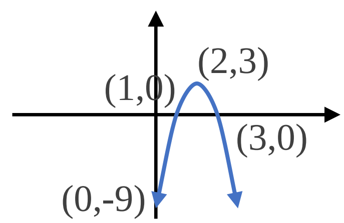

6)



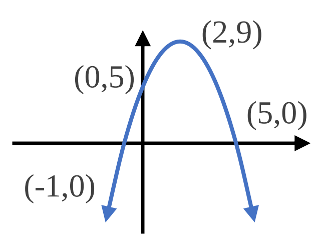

7)



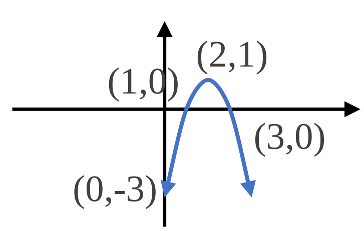

8)




9)



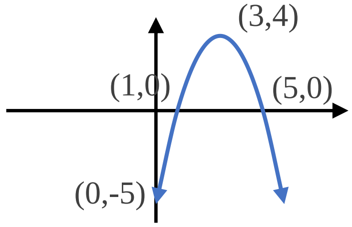

10)




*The Answers to Practice Exercises are continued on the next page.*

*ANSWERS to Practice Exercises: Section 4.4 (continued)*

11)

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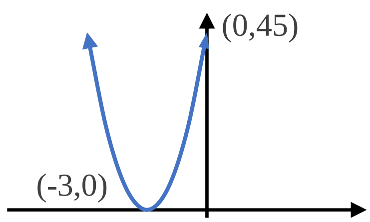


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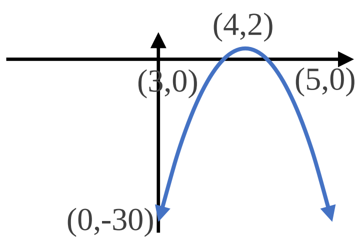


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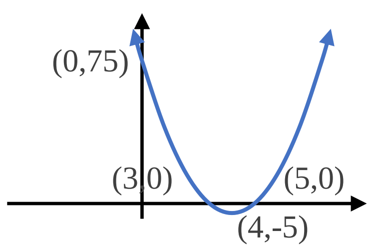


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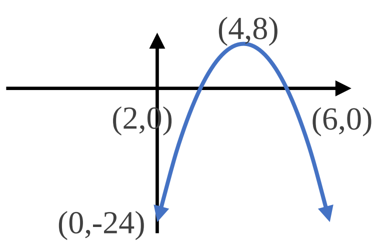


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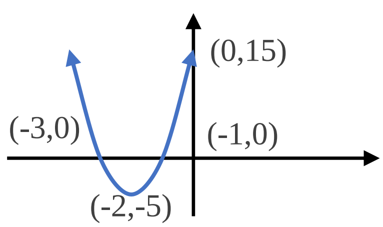


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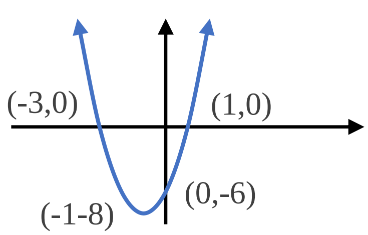


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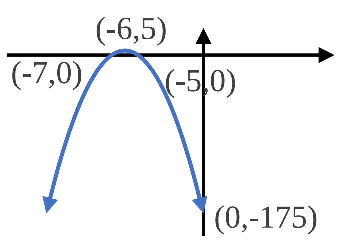


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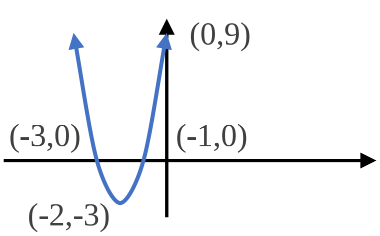


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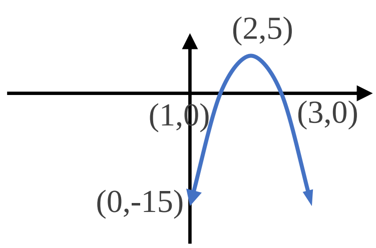


*The Answers to Practice Exercises are continued on the next page.*

*ANSWERS to Practice Exercises: Section 4.4 (continued)*

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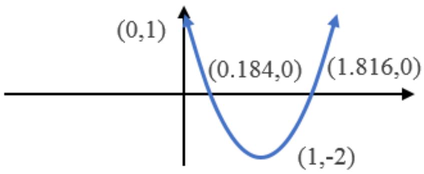


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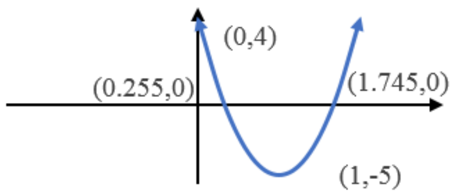


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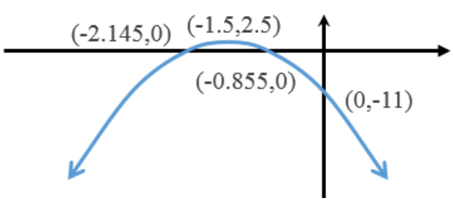


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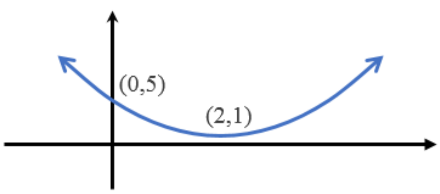


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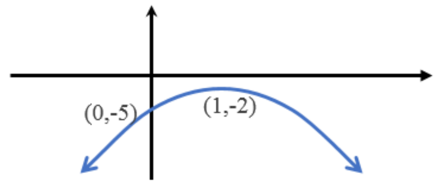


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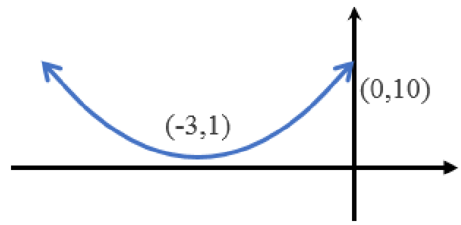


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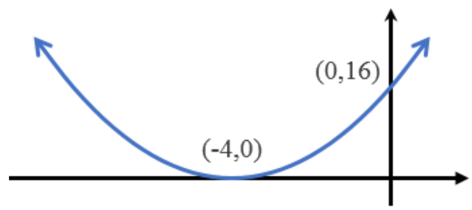


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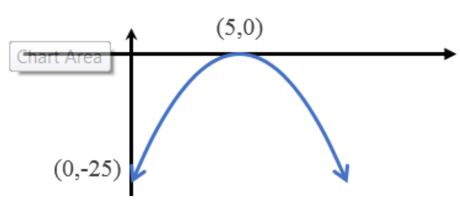


# Section 4.5: Quadratic Applications

## Objective: Solve quadratic application problems.

The vertex of the parabola formed by the graph of a quadratic equation is either a maximum point or a minimum point, depending on the sign of *a*. If *a* is a *positive* number, then the vertex is the *minimum*; if *a* is a *negative* number, then the vertex is a *maximum*.

An example of a maximum would be the highest height of a ball that has been thrown into the air. An example of a minimum would be the minimum average cost to a company for a product that has been produced.

**Example 1.** Answer each of the following questions.

Terry is on the balcony of her apartment, which is 150 feet above the ground. She tosses a ball vertically upward. The ball's height above the ground as it travels is modeled by the quadratic equation $h = -16t^2 + 64t + 150$, where $t$ is the amount of time (in seconds) the ball has been in flight and $h$ is the height of the ball (in feet) at any particular time.

a. How many seconds will it take for the ball to reach its maximum height above the ground?

| | |
|---|---|
| $h = -16t^2 + 64t + 150$ | The time $t$ is unknown; $a = -16$, $b = 64$, $c = 150$ |
| $t = \dfrac{-(64)}{2(-16)} = \dfrac{-64}{-32} = 2$ | Use $\dfrac{-b}{2a}$ to find the amount of time, $t$, that has passed when the ball reaches its maximum height |
| 2 seconds | The ball reaches its maximum height after 2 seconds |

b. What is the ball's maximum height above the ground?

| | |
|---|---|
| $h = -16t^2 + 64t + 150$ | The time that has passed, $t$, is $2$ seconds |
| $h = -16(2)^2 + 64(2) + 150$ | Substitute the value $2$ in for $t$ everywhere in the equation |
| $h = -16(4) + 64(2) + 150$ | Simplify |
| $h = -64 + 128 + 150 = 214$ | |
| $214$ feet | So, the maximum height of the ball is $214$ feet |

c. How long does it take for the ball to hit the ground? Round to the nearest tenth of a second, if necessary.

| | |
|---|---|
| $h = -16t^2 + 64t + 150$ | The time that has passed, $t$, is unknown; When the ball hits the ground, its height $h$ is zero |
| $0 = -16t^2 + 64t + 150$ | Set the quadratic equation equal to zero and solve the equation; $a = -16$, $b = 64$, $c = 150$ |
| $t = \dfrac{-(64) \pm \sqrt{(64)^2 - 4(-16)(150)}}{2(-16)}$ | Use the quadratic formula to determine the time $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| $t = \dfrac{-64 \pm \sqrt{4096 + 9600}}{-32}$ | Simplify |
| $t = \dfrac{-64 \pm \sqrt{13696}}{-32}$ | Use a calculator to evaluate each of solutions |
| $t \approx -1.7$ or $t \approx 5.7$ | Time cannot be negative; so $t \approx -1.7$ is extraneous |
| 5.7 seconds | So, the time lapsed when the ball hits the ground is approximately 5.7 seconds |

**Example 2.** Solve.

The price to charge for a product is a key business decision. If the price is low, then the business may sell many items but will not make much profit per sale. If the price is high, the business will make a large profit per sale but they will have fewer sales. Some value in the middle is "just right" and will maximize profit.

A company has determined that if they charge a price $x$, in dollars, then their profit $P$, in thousands of dollars, is given by the equation $P = -x^2 + 120x - 2000$. To maximize profit, what price should the company choose? What is the maximum profit?

| | |
|---|---|
| $P = -x^2 + 120x - 2000$ | The price, $x$, is unknown; $a = -1$, $b = 120$, $c = -2000$ |
| $x = \dfrac{-b}{2a} = \dfrac{-120}{2(-1)} = \dfrac{-120}{-2} = 60$ | Use $\frac{-b}{2a}$ to find the price $x$ that will maximize the profit. |
| $x = 60$ dollars | The company should choose \$60 as the price to maximize profit. |

| | |
|---|---|
| $P=-(60)^2+120(60)-2000$ | Substitute the value $60$ for $x$ in the equation. |
| $P=-3600+7200-2000$ | Simplify using the Order of Operations. |
| $P=1600$ | The maximum profit is 1600 thousand dollars or $1,600,000. |

**Example 3.** Solve.

Arthur sells used cell phones. He has determined that his average cost to package and ship cell phones to customers is given by the equation $C=2x^2-60x+1700$, where $x$ is the number of cell phones packaged and shipped every two weeks, and $C$ is the average cost. How many cell phones must Arthur package and ship during the two-week period in order to minimize the average cost? What is the minimum average cost?

| | |
|---|---|
| $C=2x^2-60x+1700$ | The number of cell phones, $x$, is unknown; $a=2$, $b=-60$, $c=1700$ |
| $x=-\frac{b}{2a}=-\frac{(-60)}{2(2)}=\frac{60}{4}=15$ | Use $\frac{-b}{2a}$ to find the number of cell phones that will minimize the cost. |
| $x=15$ | $15$ cell phones must be shipped to minimize the average cost. |
| $C=2(15)^2-60(15)+1700$ | Substitute the value $15$ for $x$ in the equation. |
| $C=2(15)^2-60(15)+1700$<br>$C=2(225)-60(15)+1700$<br>$C=450-900+1700$ | Simplify using the Order of Operations. |
| $C=1250$ | The minimum average cost is $1250$ dollars. |

# Practice Exercises
## Section 4.5: Quadratic Applications

**Use the following information to answer questions 1 and 2.**

George is standing on the top of a 275 foot building. He throws a ball straight up into the air. The ball's initial velocity is given as 48 ft/sec. The height $h$, in feet, of the ball after $t$ seconds is given by the equation $h=-16t^2+48t+275$.

1) How long will it take for the ball to reach its maximum height above the ground?
2) What is the maximum height that the ball reaches?

**Use the following information to answer questions 3-5.**

Shelly is standing on a platform 100 feet above the ground. She tosses a baseball straight up into the air. The equation $h=-16t^2+64t+100$ models the ball's height $h$, in feet, above the ground $t$ seconds after it was thrown.

3) How long will it take for the ball to reach its maximum height above the ground?
4) What is the maximum height that the ball reaches?
5) How many seconds does it take for the ball to finally hit the ground (rounded to the nearest tenth of a second)?

**Use the following information to answer questions 6-8.**

Les is standing on the ground. He launches a model rocket straight up into the air. The equation $h=-16t^2+64t$ models the rocket's height *h*, in feet, above the ground $t$ seconds after it was launched.

6) How long will it take for the rocket to reach its maximum height above the ground?
7) What is the maximum height that the rocket reaches?
8) How many seconds does it take for the ball to finally hit the ground (rounded to the nearest tenth of a second)?

**Use the following information to answer questions 9-10.**

The equation $P=-0.001x^2+2.45x-525$ models the profit $P$, in dollars, for $x$ lasagne meals sold each week at Mama Anna's Restaurant

9) How many lasagne meals should the restaurant sell each week in order to maximize its profit?
10) What would be the maximum weekly profit if they sell the necessary number of meals (rounded to the nearest cent)?

**Use the following information to answer questions 11-12.**

The equation $P=0.001t^2-0.24t+59.90$ closely models common stock $XYZ$'s closing price, $P$, in dollars, after $t$ days of trading on the market for the calendar year of 2015.

11) After how many days was $XYZ$ stock at its lowest value?
12) What was the stock's lowest price for 2015?

# ANSWERS to Practice Exercises
## Section 4.5: Quadratic Applications

1) 1.5 seconds
2) 311 feet
3) 2 seconds
4) 164 feet
5) 5.2 seconds
6) 2 seconds
7) 64 feet
8) 4 seconds
9) 1225 meals
10) $975.63
11) 120 days
12) $45.50

# Review: Chapter 4

**Solve the equation using the square root property.**

1) $(8s+5)^2 = 49$

2) $7(5x-6)^2 - 3 = 172$

**Solve the equation by completing the square.**

3) $5x^2 + 10x - 40 = 0$

4) $-3z^2 - 18z + 5 = -1$

5) $4p^2 - 3p - 9 = 0$

6) $m^2 + 4m - 39 = -19$

**Solve the equation by using the quadratic formula.**

7) $4m^2 - 3m - 5 = 0$

8) $-9x^2 - 3x + 2 = 0$

9) $7x^2 - 4x = -7$

10) $5x^2 = -22x - 8$

11) $3w^2 - w + 14 = 8$

12) $3x^2 - 15 = 0$

**Determine the *x*-intercept(s), *y*-intercept, and vertex of the graph of each quadratic equation.**

13) $y = -2x^2 + 24x - 40$

14) $y = 2x^2 - 20x + 32$

15) $y = x^2 - 12x + 36$

**Determine the vertex and intercepts, then use this information to sketch the graph.**

16) $y = x^2 + 6x + 8$.

17) $y = 2x^2 - 16x + 24$.

**Solve.**

18) NASA launches a rocket at $t = 0$ seconds. Its height, in meters, above sea-level in terms of time is given by the equation $h = -4.9t^2 + 58t + 241$.
   a) How high is the rocket after 8 seconds?
   b) How high was the rocket when it was initially launched?

**Solve.**

19) A large explosion causes wood and metal debris to rise vertically into the air with an initial velocity of 128 feet per second. The equation $h = 128t - 16t^2$ gives the height of the falling debris above the ground, in feet, $t$ seconds after the explosion.
    a) Use the given equation to find the height of the debris one second after the explosion.
    b) How many seconds after the explosion will the debris hit the ground?

20) If the equation $P = 4 + 5x - 2x^2$ represents the profit, in thousands of dollars, in selling $x$ thousand Bassblast speakers, how many speakers should be sold to maximize profit? What is the maximum profit?

21) We are standing on the top of a 512 feet tall building and launch a small object upward. The object's height, measured in feet, after $t$ seconds is given by the equation $h = -16t^2 + 224t + 512$.
    a) After how many seconds does the object hit the ground?
    b) What is the highest point that the object reaches?

22) We are standing on the top of a 1024 feet tall building and launch a small object upward. The object's vertical position, measured in feet, after $t$ seconds is given by the equation $h = -16t^2 + 192t + 1024$. How long does it take for the object to reach the highest point? What is the highest point that the object reaches?

# ANSWERS to Review: Chapter 4

1) $-\frac{3}{2}, \frac{1}{4}$

2) $\frac{11}{5}, \frac{1}{5}$

---

3) $2, -4$

4) $-3+\sqrt{11}, -3-\sqrt{11}$

5) $\frac{3+\sqrt{153}}{8}, \frac{3-\sqrt{153}}{8}$

6) $-2+2\sqrt{6}, -2-2\sqrt{6}$

---

7) $\frac{3+\sqrt{89}}{8}, \frac{3-\sqrt{89}}{8}$

8) $\frac{1}{3}, -\frac{2}{3}$

9) $\frac{2+3i\sqrt{5}}{7}, \frac{2-3i\sqrt{5}}{7}$

10) $-\frac{2}{5}, -4$

11) $\frac{1+i\sqrt{71}}{6}, \frac{1-i\sqrt{71}}{6}$

12) $-\sqrt{5}, \sqrt{5}$

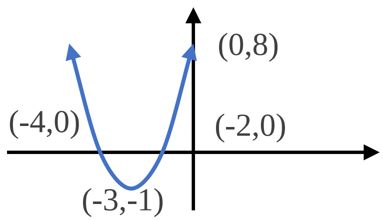
---

13) $y$-intercept: $(0,-40)$; $x$-intercepts: $(2,0)$, $(10,0)$; vertex: $(6,32)$

14) $y$-intercept: $(0,32)$; $x$-intercepts: $(2,0)$, $(8,0)$; vertex: $(5,-18)$

15) $y$-intercept: $(0,36)$; $x$-intercept: $(6,0)$; vertex: $(6,0)$

16)

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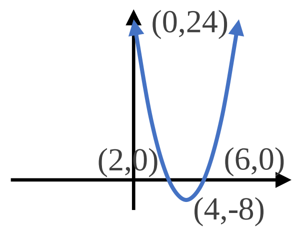


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18) 391.4 meters; 241 meters

19) 112 feet; 8 seconds

20) 1250 speakers; $7,125

21) 16 seconds; 1296 feet

22) 6 seconds; 1600 feet