Section 1.5: Factoring Special Products

Objective: Identify and factor special products including a difference of two perfect squares, perfect square trinomials, and sum and difference of two perfect cubes.

When factoring there are a few special products that, if we can recognize them, help us factor polynomials.

DIFFERENCE OF TWO PERFECT SQUARES

When multiplying special products, we found that a sum of a binomial and a difference of a binomial could multiply to a difference of two perfect squares. Here, we will use this special product to help us factor.

Difference of Two Perfect Squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Example 1. Factor completely.

$x^2 - 16$	Express each term as the square of a monomial
$= (x)^2 - (4)^2$	Apply the difference of two perfect squares formula: Here, $a = x$ and $b = 4$
$= (x+4)(x-4)$	Our Answer

Example 2. Factor completely.

$36 - y^2$	Express each term as the square of a monomial
$= (6)^2 - (y)^2$	Apply the difference of two perfect squares formula: Here, $a = 6$ and $b = y$
$= (6+y)(6-y)$	Our Answer

Example 3. Factor completely.

$9a^2 - 25b^2$	Express each term as the square of a monomial
$= (3a)^2 - (5b)^2$	Apply the difference of two perfect squares formula: Here, $a = 3a$ and $b = 5b$
$= (3a+5b)(3a-5b)$	Our Answer

PERFECT SQUARE TRINOMIAL

Another special case involves the perfect square trinomial. We had a shortcut for squaring a binomial, which can be reversed to help us factor a perfect square trinomial.

Perfect Square Trinomial:

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

If we do not recognize a perfect square trinomial at first glance, we use the *ac method*. If we get two of the same numbers, we know we have a perfect square trinomial. Then we can factor using the square roots of the first and last terms, and the sign from the middle term.

Example 4. Factor completely.

$x^2 - 6x + 9$	Multiply to 9, sum to -6 Numbers are -3 and -3, the same; a perfect square trinomial Use square roots from first and last terms and sign from middle term
$= (x-3)^2$	Our Answer

Example 5. Factor completely.

$4x^2 + 20xy + 25y^2$	Multiply to 100, sum to 20 Numbers are 10 and 10, the same; perfect square trinomial Use square roots from first and last terms and sign from middle term
$= (2x+5y)^2$	Our Answer

SUM OR DIFFERENCE OF TWO PERFECT CUBES

Another special case involves the sum or difference of two perfect cubes. The sum and the difference of two perfect cubes have very similar factoring formulas:

Sum of Two Perfect Cubes:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Difference of Two Perfect Cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Start by expressing each term as the cube of a monomial. Use these results to determine the factored form of the expression. Comparing the formulas, you may notice that the only difference is the **signs** between the terms. One way to keep these two formulas straight is to think of **SOAP**.

> **S** stands for **S**ame sign as the original polynomial. If we have a sum of two perfect cubes, we add first; if we have a difference of two perfect cubes we subtract first.
> **O** stands for **O**pposite sign. If we have a sum, then subtraction is the second sign; a difference has addition for the second sign.
> **AP** stands for **A**lways **P**ositive. The last term for both formulas has an addition sign.

The following examples demonstrate factoring the sum or difference of two perfect cubes.

Example 6. Factor completely.

$m^3 - 27$	Express each term as the cube of a monomial
$= (m)^3 - (3)^3$	Apply the difference of two perfect cubes formula $(m \quad 3)(m^2 \quad 3m \quad 9)$; Use SOAP to fill in signs
$= (m-3)\,(m^2+3m+9)$	Our Answer

Example 7. Factor completely.

$125p^3 + 8r^3$	Express each term as the cube of a monomial
$= (5p)^3 + (2r)^3$	Apply the sum of two perfect cubes formula $(5p \quad 2r)\,(25p^2 \quad 10r \quad 4r^2)$; Use SOAP to fill in signs
$= (5p+2r)\,(25p^2 - 10pr + 4r^2)$	Our Answer

The previous example illustrates an important point. When we fill in the trinomial's first and last terms, we square the monomials $5p$ and $2r$. So, our squared terms in the second set of parentheses are $5p \cdot 5p = 25p^2$ and $2r \cdot 2r = 4r^2$. Notice that when done correctly, both the number and the variable are squared. Sometimes students forget to square both the number and the variable.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial, further. As a general rule, this factor will always be *prime* (unless there is a GCF that should have been factored before applying the appropriate perfect cubes rule).

SUMMARY OF FACTORING SPECIAL PRODUCTS

The following table summarizes all of the methods that we can use to factor special products:

FACTORING SPECIAL PRODUCTS

Difference of Squares:	$a^2 - b^2 = (a+b)(a-b)$
Sum of Squares:	prime
Perfect Square Trinomial:	$a^2 + 2ab + b^2 = (a+b)^2$
	$a^2 - 2ab + b^2 = (a-b)^2$
Sum of Cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes:	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

FACTORING USING MORE THAN ONE STRATEGY

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This process is shown in the following examples.

Example 8. Factor completely.

$72x^2 - 2$	GCF is 2; factor from each term
$= 2\,(36x^2 - 1)$	Difference of two perfect squares: $36x^2 = (6x)^2$ and $1 = (1)^2$
$= 2\,(6x+1)\,(6x-1)$	Our Answer

Example 9. Factor completely.

$48x^2y - 24xy + 3y$	GCF is $3y$; factor from each term
$= 3y\,(16x^2 - 8x + 1)$	Multiply to 16, sum to -8 Numbers are -4 and -4, the same; perfect square trinomial Use square roots from first and last terms and sign from middle term
$= 3y\,(4x-1)^2$	Our Answer

Example 10. Factor completely.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$; factor from each term
$= 2ab^2(64a^3 + 27b^3)$	Sum of two perfect cubes: $64a^3 = (4a)^3$ and $27b^3 = (3b)^3$
$= 2ab^2(4a+3b)\,(16a^2 - 12ab + 9b^2)$	Our Answer

Practice Exercises
Section 1.5: Factoring Special Products

Factor completely.

1) x^2-49
2) x^2-9
3) v^2-25
4) $1-x^2$
5) p^2-4
6) $4v^2-1$
7) $64x^2-9y^2$
8) $9a^2-1$
9) $9x^2+1$
10) $3x^2-27$
11) $5n^2-20$
12) $16x^2-36$
13) $125x^2+45y^2$
14) $98a^2-50b^2$
15) $4m^2+64n^2$
16) a^2-2a+1
17) k^2+4k+4
18) x^2+6x+9
19) $n^2-8n+16$
20) x^2-6x+9
21) k^2-4k+4
22) $25p^2-10p+1$
23) x^2+2x+1
24) $25a^2+30ab+9b^2$
25) $x^2+8xy+16y^2$
26) $4a^2-20ab+25b^2$
27) $49x^2+36y^2$
28) $8x^2-24xy+18y^2$
29) $20x^2+20xy+5y^2$
30) x^3-8
31) x^3+64
32) x^3-64
33) x^3+8
34) $216-u^3$
35) $125x^3-216$
36) $125a^3-64$
37) $64x^3-27$
38) $64x^3+27y^3$
39) $32m^3-108n^3$
40) $54x^3+250y^3$

ANSWERS to Practice Exercises
Section 1.5: Factoring Special Products

1) $(x+7)(x-7)$
2) $(x+3)(x-3)$
3) $(v+5)(v-5)$
4) $(1+x)(1-x)$
5) $(p+2)(p-2)$
6) $(2v+1)(2v-1)$
7) $(8x+3y)(8x-3y)$
8) $(3a+1)(3a-1)$
9) prime
10) $3(x+3)(x-3)$
11) $5(n+2)(n-2)$
12) $4(2x+3)(2x-3)$
13) $5(25x^2+9y^2)$
14) $2(7a+5b)(7a-5b)$
15) $4(m^2+16n^2)$
16) $(a-1)^2$
17) $(k+2)^2$
18) $(x+3)^2$
19) $(n-4)^2$
20) $(x-3)^2$
21) $(k-2)^2$
22) $(5p-1)^2$
23) $(x+1)^2$
24) $(5a+3b)^2$
25) $(x+4y)^2$
26) $(2a-5b)^2$
27) prime
28) $2(2x-3y)^2$
29) $5(2x+y)^2$
30) $(x-2)(x^2+2x+4)$
31) $(x+4)(x^2-4x+16)$
32) $(x-4)(x^2+4x+16)$
33) $(x+2)(x^2-2x+4)$
34) $(6-u)(36+6u+u^2)$
35) $(5x-6)(25x^2+30x+36)$
36) $(5a-4)(25a^2+20a+16)$
37) $(4x-3)(16x^2+12x+9)$
38) $(4x+3y)(16x^2-12xy+9y^2)$
39) $4(2m-3n)(4m^2+6mn+9n^2)$
40) $2(3x+5y)(9x^2-15xy+25y^2)$