# Section 1.7: Solving Equations by Factoring

## Objective: Solve equations by factoring and using the zero product rule.

When solving linear equations such as $2x-5=21$, we can solve for the variable directly by adding $5$ and dividing by $2$ to get $13$. However, when we have $x^2$ (or a higher power of $x$), we cannot just isolate the variable as we did with the linear equations. One method that we can use to solve for the variable will require us to use a property known as the zero product rule.

> **Zero Product Rule:**
>
> **If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.**

The zero product rule tells us that if two factors are multiplied together and the result is zero, then one of the factors must be zero.

## USING THE ZERO PRODUCT RULE TO SOLVE EQUATIONS

We can use the zero product rule to help us solve equations having zero on one side and a factored expression on the other side as in the following example.

**Example 1**. Solve the equation.

$$(2x-3)(5x+1)=0 \quad \text{Set each factor equal to zero}$$

$$2x-3=0 \quad \text{or} \quad 5x+1=0 \quad \text{Solve each equation}$$

$$\underline{+3 \ \ +3} \qquad \underline{-1 \ \ -1}$$

$$\frac{2x}{2}=\frac{3}{2} \quad \text{or} \quad \frac{5x}{5}=\frac{-1}{5}$$

$$x=\frac{3}{2} \quad \text{or} \quad -\frac{1}{5} \quad \text{Our Solutions}$$

For the zero product rule to work, we must have factors to set equal to zero. This means if the expression is not already factored, we will need to factor it first, if possible.

**Example 2**. Solve the equation.

$$4x^2+x-3=0$$

Factor using the *ac* method: Find factors to multiply to $-12$ and add to $1$

Use $-3$ and $4$

$$4x^2-3x+4x-3=0$$

Factor by grouping

$$x(4x-3)+1(4x-3)=0$$

$$(4x-3)(x+1)=0$$ Set each factor equal to zero

$$4x-3=0 \quad \text{or} \quad x+1=0$$ Solve each equation

$$\underline{\quad +3 \;\; +3} \qquad \underline{\quad -1 \;\; -1}$$

$$\frac{4x}{4}=\frac{3}{4} \quad \text{or} \quad x=-1$$

$$x=\frac{3}{4} \quad \text{or} \quad -1$$ Our Solutions

Another important part of the zero product rule is that before we factor, one side of the equation must be zero. If one side of the equation is not zero, we must move terms around so that one side of the equation is zero. Generally, we like the coefficient of the $x^2$ term to be positive.

**Example 3**. Solve the equation.

$$x^2=8x-15$$ Set one side equal to 0, adding $-8x$ and $15$ to both sides of the equation

$$\underline{-8x+15 \quad -8x+15}$$

$$x^2-8x+15=0$$ Factor using the *ac method*: Find factors to multiply to $15$ and add to $-8$

Use $-5$ and $-3$

$$(x-5)(x-3)=0$$ Set each factor equal to zero

$$x-5=0 \quad \text{or} \quad x-3=0$$ Solve each equation

$$\underline{\quad +5 \;\; +5} \qquad \underline{\quad +3 \;\; +3}$$

$$x=5 \quad \text{or} \quad x=3$$

$$x=5 \quad \text{or} \quad x=3$$ Our Solutions

**Example 4**. Solve the equation.

$$(x-7)(x+3)=-9$$ Not equal to zero; multiply first using FOIL

$$x^2-7x+3x-21=-9$$ Combine like terms

$$x^2-4x-21=-9$$ Set one side equal to 0 by adding 9 to both sides of the equation

$$\underline{\qquad +9=+9}$$

$$x^2-4x-12=0$$ Factor using the *ac method*: Find factors to multiply to $-12$ and add to $-4$

Use $6$ and $-2$

$$(x-6)(x+2)=0$$ Set each factor equal to zero

$$x-6=0 \quad \text{or} \quad x+2=0$$ Solve each equation

$$\underline{\quad +6 \;\; +6} \qquad \underline{\quad -2 \;\; -2}$$

$$x=6 \quad \text{or} \quad x=-2$$ Our Solutions

**Example 5**. Solve the equation.

| | |
|---|---|
| $3x^2+4x-5=7x^2+4x-14$ | Set one side of the equation equal to 0 |
| $\underline{-3x^2-4x+5 \quad -3x^2-4x+5}$ | |
| $0=4x^2-9$ | Factor using the difference of two perfect squares |
| $0=(2x+3)(2x-3)$ | Set each factor equal to zero |
| $2x+3=0$ or $2x-3=0$ | Solve each equation |
| $\underline{-3 \quad -3}$ $\quad$ $\underline{+3 \quad +3}$ | |
| $\frac{2x}{2}=\frac{-3}{2}$ or $\frac{2x}{2}=\frac{3}{2}$ | |
| $x=-\frac{3}{2}$ or $x=\frac{3}{2}$ | Our Solutions |

Most quadratic equations (equations where the highest exponent of the variable is 2) have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

**Example 6**. Solve the equation.

| | |
|---|---|
| $4x^2=12x-9$ | Set one side of the equation equal to 0 |
| $\underline{-12x+9 \quad -12x+9}$ | |
| $4x^2-12x+9=0$ | Factor using the *ac method*: Find factors to multiply to 36 and add to $-12$ |
| | Use $-6$ and $-6$ (the same); a perfect square trinomial |
| $(2x-3)^2=0$ | Since $(2x-3)^2=(2x-3)(2x-3)$, both factors are identical. Set the factor $(2x-3)$ equal to zero. |
| $2x-3=0$ | Solve the equation |
| $\underline{+3 \quad +3}$ | |
| $\frac{2x}{2}=\frac{3}{2}$ | |
| $x=\frac{3}{2}$ | Our Solution |

As always, it will be important to factor out the GCF first if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This might give us more than just two solutions. The next few examples illustrate this.

**Example 7**. Solve the equation.

| | |
|---|---|
| $4x^2=8x$ | Set equal to 0 by subtracting $8x$ from both sides |
| $\underline{-8x \quad -8x}$ | |
| $4x^2-8x=0$ | Factor the GCF of $4x$ |

$$4x(x-2)=0$$ Set each factor equal to zero

$$\frac{4x}{4}=\frac{0}{4} \quad \text{or} \quad x-2=0$$ Solve each equation

$$x=0 \quad \text{or} \quad \underline{\;+2 \quad +2\;}$$
$$x=2$$

$$x=0 \quad \text{or} \quad x=2$$ Our Solutions

**Example 8**. Solve the equation.

$$2x^3-14x^2+24x=0$$ Factor the GCF of $2x$

$$2x(x^2-7x+12)=0$$ Factor with *ac method*: Find factors to multiply to 12 and add to $-7$

Use $-3$ and $-4$

$$2x(x-3)(x-4)=0$$ Set each factor equal to zero

$$\frac{2x}{2}=\frac{0}{2} \quad \text{or} \quad x-3=0 \quad \text{or} \quad x-4=0$$ Solve each equation

$$\underline{\;+3 \quad +3\;} \qquad \underline{\;+4 \quad +4\;}$$

$$x=0 \qquad x=3 \qquad x=4$$

$$x=0,\ 3, \text{or } 4$$ Our Solutions

**Example 9**. Solve the equation.

$$6x^2+21x-27=0$$ Factor the GCF of $3$

$$3(2x^2+7x-9)=0$$ Factor with *ac method*: Find factors to multiply to $-18$ and add to $7$

Use $9$ and $-2$

$$3(2x^2+9x-2x-9)=0$$ Factor by grouping

$$3[x(2x+9)-1(2x+9)]=0$$

$$3(2x+9)(x-1)=0$$ Set each factor equal to zero

$$3=0 \quad \text{or} \quad 2x+9=0 \quad \text{or} \quad x-1=0$$ Solve each equation

$$\underline{\;-9=-9\;} \qquad \underline{\;+1 \quad +1\;}$$

$$3\neq 0 \quad \text{or} \quad \frac{2x}{2}=\frac{-9}{2} \quad \text{or} \quad x=1$$

$$x=-\frac{9}{2}$$

$$x=-\frac{9}{2} \quad \text{or} \quad x=1$$ Our Solutions

In the previous example, the GCF did not contain a variable. When we set this factor equal to zero, we get a false statement. No solution comes from this factor. When the GCF has no variables, we may skip setting the GCF equal to zero.

Just as not all polynomials can be factored, not all equations can be solved by factoring. If an equation cannot be solved by factoring, we will have to use another method. These other methods are saved for another lesson.

## APPLICATIONS OF SOLVING EQUATIONS

In science, we often use a mathematical model to describe a physical situation. To answer questions about the situation, we may need to set up and solve an equation. In this section, we will be able to solve the equations by factoring.

**Example 10**. Bob is on the balcony of his apartment, which is 80 feet above the ground. He tosses a ball vertically upward. The ball's height above the ground as it travels is modeled by the equation $h = -16t^2 + 64t + 80$ where $t$ is the time (in seconds) the ball has been in flight and $h$ is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

| | |
|---|---|
| $h = -16t^2 + 64t + 80$ | The time that has passed, $t$, is unknown; when the ball hits the ground, its height $h$ is zero |
| $0 = -16t^2 + 64t + 80$ | Set the quadratic equation equal to zero and solve by factoring |
| $0 = -16(t^2 - 4t - 5)$ | Factor the GCF of $-16$ |
| $0 = -16(t-5)(t+1)$ | Factor with *ac method:* Find factors to multiply to $-5$ and add to $-4$<br>Use $-5$ and 1 |
| $-16 = 0$ or $t - 5 = 0$ or $t + 1 = 0$ | Set each factor equal to zero |
| $+5 \ +5$     $-1 \ -1$ | |
| $-16 \neq 0$ or $t = 5$ or $t = -1$ | |
| $t = 5$ or $t = -1$ | Time cannot be negative; so $t = -1$ extraneous (not considered to be a solution to the equation in this context. |
| The ball hits the ground after 5 seconds. | Our Solution |

# Practice Exercises
## Section 1.7: Solving Equations by Factoring

**Solve each equation by factoring.**

1) $(x-1)(x+4)=0$
2) $0=(2x+5)(x-7)$
3) $x^2-4=0$
4) $2x^2-18x=0$
5) $6x^2-150=0$
6) $p^2+4p-32=0$
7) $2n^2+10n-28=0$
8) $m^2-m-30=0$
9) $7x^3+26x^2+15x=0$
10) $-16t^2+24t+16=0$
11) $x^2-4x-8=-8$
12) $x^2-5x-1=-5$
13) $a^3-6a^2+6a=-2a$
14) $7x^2+17x-20=-8$
15) $4n^2-13n+8=5$
16) $7r^2+84=-49r$
17) $x^2-6x=-9$
18) $7n^2-28n=0$
19) $3v^2+7v=40$
20) $6b^2=5+7b$
21) $35x^2+120x=-45$
22) $3n^2+3n=6$
23) $k^2+24k+89=6k+8$
24) $a^2+7a-9=-3+6a$
25) $9x^2-46+7x=7x+8x^2+3$
26) $x^2+10x+30=6$
27) $2m^2+19m+40=-2m$
28) $5n^2+41n+40=-2$
29) $24x^2+11x-80=3x$
30) $121w^2+8w-7=8w-6$

**Solve each application.**

31) A ball is tossed vertically upward from a building which is 96 feet above the ground. The ball's height above the ground as it travels is modeled by the equation $h=-16t^2+16t+96$ where $t$ is the time (in seconds) the ball has been in flight and $h$ is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

32) An explosion causes debris to fly vertically upward with an initial speed of 80 feet per second. The height of the debris above the ground is modeled by the equation $h=-16t^2+80t$ where $t$ is the time (in seconds) after the explosion and $h$ is the height of the debris (in feet) at any particular time. How long does it take for the debris to hit the ground?

# ANSWERS to Practice Exercises
## Section 1.7: Solving Equations by Factoring

1) $1, -4$
2) $-\frac{5}{2}, 7$
3) $-2, 2$
4) $0, 9$
5) $-5, 5$
6) $4, -8$
7) $2, -7$
8) $-5, 6$
9) $-\frac{5}{7}, -3, 0$
10) $-\frac{1}{2}, 2$
11) $4, 0$
12) $1, 4$
13) $0, 4, 2$
14) $\frac{4}{7}, -3$
15) $\frac{1}{4}, 3$
16) $-4, -3$
17) $3$
18) $4, 0$
19) $\frac{8}{3}, -5$
20) $-\frac{1}{2}, \frac{5}{3}$
21) $-\frac{3}{7}, -3$
22) $-2, 1$
23) $-9$
24) $2, -3$
25) $-7, 7$
26) $-4, -6$
27) $-\frac{5}{2}, -8$
28) $-\frac{6}{5}, -7$
29) $\frac{5}{3}, -2$
30) $-\frac{1}{11}, \frac{1}{11}$

31) The ball hits the ground after 3 seconds.

32) The debris hits the ground after 5 seconds.