Section 2.6: Solving Rational Equations

Objective: Solve rational equations by identifying and multiplying by the least common denominator.

When solving equations that are made up of rational expressions, we will use the same strategy we use to solve linear equations with fractions. As shown in the example below, we clear the fractions by multiplying both sides of the equation by the least common denominator.

Example 1. Solve the equation.

$\frac{2}{3}x - \frac{5}{6} = \frac{3}{4}$	Multiply each term by the LCD 12
$\frac{2(12)}{3}x - \frac{5(12)}{6} = \frac{3(12)}{4}$	Reduce fractions, which clears the denominators $\frac{2(12)}{3}x = 8x;\ -\frac{5(12)}{6} = -10;\ \frac{3(12)}{4} = 9$
$8x - 10 = 9$	Solve for x
$+10 \quad +10$	First, add 10 to both sides
$\frac{8x}{8} = \frac{19}{8}$	Then; divide both sides by 8
$x = \frac{19}{8}$	Our Solution

To solve rational equations, we will use the same process of multiplying by the LCD to clear the fractions. We will also have to be aware of values of the variable that must be excluded. Recall that whenever its denominator is zero, a fraction is undefined; so, any solution that would make the denominator's value zero is not a part of the solution set. For this reason, we will always check our proposed solutions to make sure they are not excluded values.

SOLVING RATIONAL EQUATIONS

1. Determine the excluded value(s), the value(s) of the variable making a denominator zero.
2. Multiply both sides of the equation by the LCD to clear the fractions.
3. Solve the resulting equation.
4. Reject any proposed solution(s) that is an excluded value.

Example 2. Solve the equation.

$$\frac{5x+5}{x+2}+3x=\frac{x^2}{x+2}$$

Find values of x for which denominators are 0:
$x+2=0$ when $x=-2$
So $x=-2$ is an excluded value

Multiply each term by the LCD $(x+2)$

$$\frac{(5x+5)(x+2)}{x+2}+3x(x+2)=\frac{x^2(x+2)}{x+2}$$

Reduce fractions

$$5x+5+3x(x+2)=x^2$$

Distribute

$$5x+5+3x^2+6x=x^2$$

Combine like terms

$$\begin{array}{l} 3x^2+11x+5=x^2 \\ \underline{-x^2 \qquad\qquad\quad -x^2} \end{array}$$

Make equation equal zero by subtracting x^2 from both sides

$$2x^2+11x+5=0$$

Factor completely

$$(2x+1)(x+5)=0$$

Set each factor equal to zero

$$\begin{array}{lcl} 2x+1=0 & \text{or} & x+5=0 \\ \underline{\;-1 \quad -1} & & \underline{\;-5 \quad -5} \\ \frac{2x}{2}=-\frac{1}{2} & \text{or} & x=-5 \end{array}$$

Solve each equation

$$x=-\frac{1}{2} \text{ or } -5$$

Proposed solutions do not match the excluded value $x=-2$ so these are our solutions

The LCD can contain several factors in these problems. Even as the LCD gets more complex, the process for solving a rational equation is the same.

Example 3. Solve the equation.

$$\frac{x}{x+2}+\frac{1}{x+1}=\frac{5}{(x+1)(x+2)}$$

Find values of x for which denominators are 0:
$x+2=0$ when $x=-2$
$x+1=0$ when $x=-1$
So $x=-2$ and $x=-1$ are excluded values

Use factors to find the LCD $(x+1)(x+2)$

Multiply each term by the LCD $(x+1)(x+2)$

$$\frac{x(x+1)(x+2)}{x+2}+\frac{1(x+1)(x+2)}{x+1}=\frac{5(x+1)(x+2)}{(x+1)(x+2)}$$ Reduce fractions

$$x(x+1)+1(x+2)=5$$ Distribute

$$x^2+x+x+2=5$$ Combine like terms

$$\begin{array}{r} x^2+2x+2=5 \\ -5 \quad -5 \\ \hline \end{array}$$ Make equation equal zero by subtracting 5 from both sides

$$x^2+2x-3=0$$ Factor completely

$$(x+3)(x-1)=0$$ Set each factor equal to zero

$$\begin{array}{rcr} x+3=0 & \text{or} & x-1=0 \\ -3 \quad -3 & & +1 \quad +1 \\ \hline x=-3 & \text{or} & x=1 \end{array}$$ Solve each equation

$x=-3$ or $x=1$ Proposed solutions do not match the excluded values $x=-2$ and $x=-1$ so these are our solutions

In the previous example the denominators were already in factored form. More often, we will need to factor before finding the LCD.

Example 4. Solve the equation.

$$\frac{x}{x-1}-\frac{1}{x-2}=\frac{11}{x^2-3x+2}$$ Factor denominator

$$\frac{x}{x-1}-\frac{1}{x-2}=\frac{11}{(x-1)(x-2)}$$

Find values of x for which denominators are 0:

$x-1=0$ when $x=1$

$x-2=0$ when $x=2$

So $x=1$ and $x=2$ are excluded values

Use factors to find LCD: $(x-1)(x-2)$

Multiply each term by the LCD $(x-1)(x-2)$

$$\frac{x(x-1)(x-2)}{x-1}-\frac{1(x-1)(x-2)}{x-2}=\frac{11(x-1)(x-2)}{(x-1)(x-2)}$$ Reduce fractions

$$x(x-2)-1(x-1)=11$$ Distribute

$$x^2-2x-x+1=11$$ Combine like terms

$$\begin{array}{r} x^2-3x+1=11 \\ -11 \quad -11 \\ \hline x^2-3x-10=0 \end{array}$$ Make equation equal zero by subtracting 11 from both sides

Factor completely

$$(x-5)(x+2)=0$$ Set each factor equal to zero

$$\begin{array}{rcr} x-5=0 & \text{or} & x+2=0 \\ +5 \quad +5 & & -2 \quad -2 \\ \hline x=5 & \text{or} & x=-2 \end{array}$$ Solve each equation

$$x=5 \text{ or } -2$$ Proposed solutions do not match the excluded values $x=1$ and $x=2$ so these are our solutions

If we are subtracting a fraction with more than one term in the numerator, it may be easier to avoid future sign errors by first distributing the negative throughout the numerator in the subtrahend (rational expression immediately following the subtraction sign).

Example 5. Solve the equation.

$$\frac{x-2}{x-3}-\frac{x+2}{x+2}=\frac{5}{8}$$ Distribute the negative through numerator

$$\frac{x-2}{x-3}+\frac{-x-2}{x+2}=\frac{5}{8}$$ Find values of x for which denominators are 0:

$x-3=0$ when $x=3$

$x+2=0$ when $x=-2$

So $x=3$ and $x=-2$ are excluded values

Use factors to find LCD: $8(x-3)(x+2)$

Multiply each term by the LCD $8(x-3)(x+2)$

$$\frac{(x-2)8(x-3)(x+2)}{x-3}+\frac{(-x-2)8(x-3)(x+2)}{x+2}=\frac{5\cdot 8(x-3)(x+2)}{8}$$ Reduce fractions

$$8(x-2)(x+2)+8(-x-2)(x-3)=5(x-3)(x+2)$$ Multiply (FOIL)

$$8(x^2-4)+8(-x^2+x+6)=5(x^2-x-6)$$ Distribute

$$8x^2-32-8x^2+8x+48=5x^2-5x-30$$ Combine like terms

$$\begin{array}{rcl} 8x+16 &=& 5x^2-5x-30 \\ \underline{-8x-16} & & \underline{-8x-16} \\ 0 &=& 5x^2-13x-46 \end{array}$$

Make equation equal zero by subtracting $8x$ and 16

Factor completely using *ac method*

$$0=(5x-23)(x+2)$$ Set each factor equal to zero

$$\begin{array}{ccc} 5x-23=0 & \text{or} & x+2=0 \\ \underline{+23\quad +23} & & \underline{-2\quad -2} \\ \frac{5x}{5}=\frac{23}{5} & \text{or} & x=-2 \end{array}$$

Solve each equation

$$x=\frac{23}{5}\text{ or }-2$$

The proposed solution $x=\frac{23}{5}$ *is* a solution because it is not an excluded value.

The proposed solution $x=-2$ is *not* a solution because it is an excluded value.

$$x=\frac{23}{5}$$ Our Solution

In the previous example, one of the proposed solutions was an excluded value because it made one of the denominators zero. When this happens, we exclude this result. The only solution(s) to the original rational equation are those that do not result with zero in the denominator.

In Examples 2 through 5 above, the equation that resulted after clearing the fractions was a quadratic equation. Notice in Example 6 below that the resulting equation is a linear equation.

Example 6. Solve the equation.

$\dfrac{2}{x-2}=\dfrac{x}{x-2}-3$	Find values of x for which denominators are 0: $x-2=0$ when $x=2$ So $x=2$ is an excluded value Use factors to find LCD: $(x-2)$ Multiply each term by the LCD $(x-2)$
$\dfrac{2(x-2)}{x-2}=\dfrac{x(x-2)}{x-2}-3(x-2)$	Reduce fractions
$2=x-3(x-2)$	Distribute
$2=x-3x+6$	Combine like terms
$2=-2x+6$ $-6 \qquad -6$	Solve for x by subtracting 6
$\dfrac{-4}{-2}=\dfrac{-2x}{-2}$ $2=x$	Divide by -2
No solution	The proposed solution $x=2$ is *not* a solution because it is an excluded value. Since there are no other proposed solutions, the original equation has no solution.

Practice Exercises
Section 2.6: Solving Rational Equations

Solve each equation:

1) $\frac{x}{2}-\frac{x}{3}=4$

2) $\frac{3}{x}+\frac{1}{2}=\frac{7}{x}$

3) $\frac{1}{3x}=\frac{5}{6}+\frac{3}{8x}$

4) $\frac{3}{x+8}=\frac{4}{6-x}$

5) $\frac{x+1}{x-1}=\frac{2x+3}{2x-5}$

6) $\frac{4}{x+3}+\frac{5}{x-1}=\frac{-7}{x^2+2x-3}$

7) $\frac{7}{x-4}-\frac{3}{x+4}=\frac{22}{x^2-16}$

8) $\frac{x+7}{x+3}=\frac{4}{x+3}$

9) $\frac{3m}{2m-5}-\frac{7}{3m+1}=\frac{3}{2}$

10) $\frac{4x}{2x-6}-\frac{4}{5x-15}=\frac{1}{2}$

11) $\frac{4-x}{1-x}=\frac{12}{3-x}$

12) $\frac{1}{x-5}+\frac{1}{x+5}=\frac{10}{x^2-25}$

13) $\frac{7}{y-3}-\frac{1}{2}=\frac{y-2}{y-4}$

14) $\frac{2}{3-x}-\frac{6}{8-x}=1$

15) $\frac{1}{x+2}-\frac{1}{2-x}=\frac{3x+8}{x^2-4}$

16) $\frac{x+2}{3x-1}-\frac{1}{x}=\frac{3x-3}{3x^2-x}$

17) $\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{5}{6}$

18) $\frac{x-1}{x-3}+\frac{x+2}{x+3}=\frac{3}{4}$

19) $\frac{3}{2x+1}+\frac{2x+1}{1-2x}=1-\frac{8x^2}{4x^2-1}$

20) $\frac{3x-5}{5x-5}+\frac{5x-1}{7x-7}-\frac{x-4}{1-x}=2$

21) $\frac{x-2}{x+3}-\frac{1}{x-2}=\frac{1}{x^2+x-6}$

22) $\frac{x-1}{x-2}+\frac{x+4}{2x+1}=\frac{1}{2x^2-3x-2}$

23) $\frac{3}{x+2}+\frac{x-1}{x+5}=\frac{5x+20}{6x+24}$

24) $\frac{x}{x+3}-\frac{4}{x-2}=\frac{-5x^2}{x^2+x-6}$

25) $\frac{x}{x-1}-\frac{2}{x+1}=\frac{4x^2}{x^2-1}$

26) $\frac{2x}{x+2}+\frac{2}{x-4}=\frac{3x}{x^2-2x-8}$

27) $\frac{2x}{x+1}-\frac{3}{x+5}=\frac{-8x^2}{x^2+6x+5}$

28) $\frac{x}{x+1}-\frac{3}{x+3}=\frac{-2x^2}{x^2+4x+3}$

29) $\frac{x-5}{x-9}+\frac{x+3}{x-3}=\frac{-4x^2}{x^2-12x+27}$

30) $\frac{x-3}{x+6}+\frac{x-2}{x-3}=\frac{x^2}{x^2+3x-18}$

ANSWERS to Practice Exercises
Section 2.6: Solving Rational Equations

1) 24
2) 8
3) $-\frac{1}{20}$
4) -2
5) $-\frac{1}{2}$
6) -2
7) $-\frac{9}{2}$
8) no solution
9) -5
10) $-\frac{7}{15}$
11) $-5, 0$
12) no solution
13) $\frac{16}{3}, 5$
14) $2, 13$
15) -8
16) 2
17) $-\frac{1}{5}, 5$
18) $-\frac{9}{5}, 1$
19) $\frac{3}{2}$
20) 10
21) $0, 5$
22) $-2, \frac{5}{3}$
23) $4, 7$
24) -1
25) $\frac{2}{3}$
26) $\frac{1}{2}$
27) $\frac{3}{10}$
28) 1
29) $-\frac{2}{3}$
30) -1