Section 2.8: Variation

Objective: Model and solve direct, inverse, joint and combined variation problems.

Variation problems are used to show the relationship between quantities. In this section, we will examine how quantities can vary directly, inversely and jointly, as well as explore what happens when more than one relationship must be considered within a problem.

Each type of variation problem will require that we first find the constant of variation k. Once that constant has been established, the relationship is defined and specific problems can be solved.

DIRECT VARIATION

When quantities vary directly, we say that "*y varies directly as x*" or that "*y varies directly in proportion to x*". Direct variation problems are modeled using linear equations in the form of $y = kx$.

DIRECT VARIATION

$$y = kx$$

We first find the value of the constant of variation k; then, we solve for a specific quantity.

Example 1. Solve the variation problem.

Given: y varies directly as x, and $y = 80$ when $x = 20$. Find y when $x = 65$.

$80 = k(20)$	Use the direct variation model $y = kx$; Substitute the given values for x and y, and solve for k.
$\frac{80}{20} = \frac{k(20)}{20}$	Divide both sides by 20.
$4 = k$	The constant of variation k is 4.
$y = 4(65)$	Substitute $k = 4$ and $x = 65$ into $y = kx$; multiply to find the value of y.
$y = 260$	Our Solution

Example 2. Solve the variation problem.

The dosage of a medication that is prescribed by a doctor varies directly as the weight of the patient. A doctor prescribes 2.5 milliliters (ml) of a medication for a 200-pound patient. How many milliliters of this medication can be prescribed for someone who weighs 220 pounds?

$2.5 = k(2000)$	The dosage y varies directly as the weight x of the patient. Use the direct variation model $y = kx$. Substitute the given values $y = 2.5$ ml and $x = 200$ pounds, and solve for k.
$\frac{2.5}{200} = \frac{k(200)}{200}$	Divide both sides by 200.
$0.0125 = k$	The constant of variation k is 0.0125.
$y = 0.0125(220)$	Substitute $k = 0.0125$ and $x = 220$ into $y = kx$; multiply to find the value of y.
$y = 2.75\,\text{ml}$	Our Solution

INVERSE VARIATION

When quantities vary inversely, we say that "y varies inversely as x" or that "y varies inversely in proportion to x". Inverse variation problems are modeled using rational equations in the form of $y = \frac{k}{x}$.

INVERSE VARIATION

$$y = \frac{k}{x}$$

Example 3. Solve the variation problem.

Given: y varies inversely as x, and $y = 95$ when $x = 3$. Find y when $x = 15$.

$95 = \frac{k}{3}$	Use the inverse variation model $y = \frac{k}{x}$. Substitute the given values for x and y, and solve for k.

$(95)(3)=\left(\frac{k}{3}\right)\left(\frac{3}{1}\right)$ Multiply both sides by 3.

$k=285$ The constant of variation k is 285.

$y=\frac{285}{15}$ Substitute $k=285$ and $x=15$ into $y=\frac{k}{x}$; divide to find the value of y.

$y=19$ Our Solution

Example 4. Solve the variation problem.

The frequency of a vibrating string varies inversely with its length. A vibrating string of length 40 inches has a frequency of 440 hertz. What is the frequency of a string of length 20 inches?

$440=\frac{k}{40}$ The frequency y varies inversely as the length x. Use the inverse variation model. $y=\frac{k}{x}$.
Substitute the given values for x and y, and solve for k.

$(440)(40)=\left(\frac{k}{40}\right)\left(\frac{40}{1}\right)$ Multiply both sides by 40.

$17,600=k$ The constant of variation k is $17,600$.

$y=\frac{17,600}{20}$ Substitute $k=17,600$ and $x=20$ into $y=\frac{k}{x}$; divide to find the value of y.

$y=880$ hertz Our Solution

JOINT VARIATION

When quantities vary jointly, we say that "y varies directly as the product of two or more variables" or that "y varies directly in proportion to two or more variables". Joint variation problems are modeled using equations in the form of $y=kxz$.

JOINT VARIATION

$$y = kxz$$

Example 5. Solve the variation problem.

Given: y varies jointly as the square of x and z, and $y = 1334$ when $x = 8$ and $z = 3$. Find y when $x = 5$ and $z = 6$.

$1344 = k(8)^2(3)$	Use the joint variation model $y = kxz$. In this case, we have the square of x so use the form $y = kx^2z$. Substitute the given values for x, y, and z, and solve for k.
$1344 = k(64)(3)$	Use order of operations to simplify.
$1344 = k(192)$	Divide both sides by 192.
$k = 7$	The constant of variation k is 7.
$y = 7(5)^2(6)$	Substitute $k = 7$, $x = 5$, and $z = 6$ into $y = kx^2z$; simplify.
$y = 1050$	Our Solution

Example 6. Solve the variation problem.

The volume (V) of a closed box with fixed height varies jointly as the length (L) of its base and the width (W) of its base. A box whose base's length is 40 inches and width is 5 inches has a volume of 700 cubic inches. What is the volume of a box whose base's length is 35 inches and whose width is 4 inches?

$700 = k(40)(5)$	The volume of the box V varies jointly as the length L and width W of its base. Use the joint variation model written as $V = kLW$. Substitute the given values for V, L and W, and solve for k.
$700 = k(200)$	Multiply $(40)(50)$.

$\frac{700}{200}=\frac{k(200)}{200}$	Divide both sides by 200.
$3.5=k$	The constant of variation k is 3.5.
$V=3.5\,(35)\,(4)$	Substitute $k=3.5$, $L=35$, and $W=4$ into $V=kLW$; multiply to find V.
$V=490$ cubic inches	Our Solution

COMBINED VARIATION

Sometimes direct, inverse, and/or joint variation occurs within the same situation. When quantities vary in more than one way, we include all relationships in one equation. The examples below illustrate combined variation.

Example 7. Solve the variation problem.

Given: a varies directly as b and inversely as c raised to the fourth power, and $a=5$ when $b=10$ and $c=2$. Find a when $b=7$ and $c=1$.

$5=\frac{k(10)}{2^4}$	Start with $a=kb$ as the variation model since a varies directly as b.
	We must also divide by c^4 since a varies inversely as c^4. So the combined variation model is $a=\frac{kb}{c^4}$.
	Substitute the given values for a, b and c, and solve for k.
$5=\frac{k(10)}{16}$	Use order of operations to simplify:
$5(16)=\frac{k(10)}{16}(16)$	Multiply both sides by 16.
$80=k(10)$	Divide both sides by 10.
$k=8$	The constant of variation k is 8.

$a = \frac{(8)(7)}{1^4}$ Substitute $k = 8$, $b = 7$, and $c = 1$ into $a = \frac{kb}{c^4}$; evaluate to find a.

$a = 56$ Our Solution

Example 8. Solve the variation problem.

A person's intelligence quotient (IQ) varies directly as a person's mental age (m) and inversely as their chronological age (c). A person whose mental age is 35 and whose chronological age is 25 has an IQ of 140. What is the IQ of a 70 year old person whose mental age is 77?

$140 = \frac{k(35)}{25}$ Start with $IQ = km$ as the variation model since IQ varies directly as m. We must also divide by c since IQ varies inversely as c. So the variation equation is modeled by $IQ = \frac{km}{c}$. Substitute the given values for IQ, m and c, and solve for k.

$140(25) = \frac{k(35)}{25}(25)$ Multiply both sides by 25.

$\frac{3500}{35} = \frac{k(35)}{35}$ Divide both sides by 35.

$100 = k$ The constant of variation, k, is 100.

$IQ = \frac{100(77)}{70}$ Substitute $k = 100$, $m = 77$, and $c = 70$ into $IQ = \frac{km}{c}$; evaluate to find IQ.

$IQ = 110$ Our Solution

Practice Exercises
Section 2.8: Variation

Solve each variation problem.

1) y varies directly as x. When $x = 15, y = 9$. What is the value of y when $x = 6.3$?

2) y varies directly as x. When $x = 80, y = 65$. What is the value of y when $x = 72$?

3) y varies directly as x. When $x = 15, y = 540$. What is the value of y when $x = 9.75$?

4) y varies directly as x. When $x = 4.8, y = 25.2$. What is the value of y when $x = 7.4$?

5) y varies inversely as x. When $x = 65, y = 9$. What is the value of y when $x = 50$?

6) y varies inversely as x. When $x = 12, y = 5$. What is the value of y when $x = 4$?

7) y varies inversely as x. When $x = 9.2, y = 2.5$. What is the value of y when $x = 4.6$?

8) y varies inversely as x. When $x = 3.2, y = 70$. What is the value of y when $x = 8$?

9) z varies directly as x and inversely as the square of y. When $x = 3$ and $y = 4$, $z = 12$. What is the value of z when $x = 12$ and $y = 8$?

10) The force, F, needed to stretch a spring varies directly as a certain distance, x, where k is the spring constant measured in Newtons/cm. If a 24 Newton force stretches a certain spring by 20 cm, determining the force needed to stretch the spring 36 cm.

11) Jean wants to invest in property in order to build and sell houses. In the area where she would like to make her purchase, she has found that zoning regulations dictate that the amount of land is directly proportional to the number of houses that can be built on that land. She was able to determine that on 3.42 acres of land, 6 houses can be built. She wishes to build 10 houses. How much land does she need?

12) The time that it takes to fill a swimming pool with water varies directly as the depth of the pool. A swimming pool manufacturer claims that a backyard swimming pool that is 4 feet deep can be filled with water in 3 hours. If this is true, how long would it take to fill a pool that is 8 feet deep?

The Practice Exercises are continued on the next page.

Practice Exercises: Section 2.8 (continued)

13) There is a direct relationship between the number of hours spent working on a project and the grade a student receives for that project. Students who spend 2.5 hours on a project earn an average of 75 points on that project. How many points should Greg earn if he spends 3 hours on his project?

14) The frequency of a vibrating piano string varies inversely to its length. An 18-inch piano string vibrates at 336 cycles/second. What is the frequency of a vibrating 21-inch piano string?

15) Entrepreneurs know that number of items sold varies inversely with the price of their product. Alice produces natural chew toys for dogs. She knows that she can sell 250 toys per month when she charges $6 per toy. How many toys should she expect to sell if she charges $4 per toy?

16) The cost per person to rent an airport limousine for one ride is inversely proportional to the number of passengers in the vehicle. When five people rent the airport limousine, each person pays $50. How much would each person pay if eight people rent the airport limousine?

17) Kinetic energy varies jointly with mass and the square of velocity. If the mass is 20 kilograms and the velocity is 3 meters per second, the kinetic energy is 90 Joules. What is the kinetic energy if the mass is 10 kilograms and the velocity is 5 meters per second?

18) The volume of a pyramid is jointly proportional to the area of its base and to its height. A pyramid with base area 40 square meters and height 6 meters has a volume of 80 cubic meters. What is the volume of a pyramid with base area 30 square meters and height 8 meters?

ANSWERS to Practice Exercises
Section 2.8: Variation

1) 3.78

2) 58.5

3) 351

4) 38.85

5) 11.7

6) 15

7) 5

8) 28

9) 12

10) 43.2 Newtons

11) 5.7 acres

12) 6 hours

13) 90 points

14) 288 cycles/second

15) 375 toys

16) $31.25

17) 125 Joules

18) 80 cubic meters