Section 3.2: Higher Roots

Objective: Simplify radicals with an index greater than two.

While square roots are the most common type of radical we work with, we can take higher roots of numbers as well: cube roots, fourth roots, fifth roots, etc. Following is a definition of higher roots.

> **n^{th} ROOT**
>
> For a positive integer $n > 1$, the principal n^{th} root of a is the number b such that $b^n = a$.
> We write $b = \sqrt[n]{a}$.
>
> NOTE: If *n* is even, *a* and *b* are *nonnegative*.

We call b the n^{th} root of a. The small letter n is called the **index**. It tells us which root we are taking, or which power we are "un-doing". For square roots, the index is 2. As this is the most common root, the two is not usually written.

The following example includes several higher roots.

Example 1. Evaluate.

$\sqrt[3]{125} = 5$ because $5^3 = 125$	$\sqrt[3]{-64} = -4$ because $(-4)^3 = -64$
$\sqrt[3]{8} = 2$ because $2^3 = 8$	$\sqrt[7]{-128} = -2$ because $(-2)^7 = -128$
$\sqrt[4]{81} = 3$ because $3^4 = 81$	$\sqrt[4]{16} = 2$ because $2^4 = 16$
$\sqrt[5]{32} = 2$ because $2^5 = 32$	$\sqrt[4]{-16}$ is not a real number

We must be careful of a few things as we work with higher roots. First, it is important to check the index on the root. For example, $\sqrt{81} = 9$ because $9^2 = 81$ but $\sqrt[4]{81} = 3$ because $3^4 = 81$. Another thing to watch out for is negative numbers in the radicand. We can take an odd root of a negative number because a negative number raised to an odd power is still negative. However, the even root of a negative number is not a real number. In a later section we will discuss how to work with even roots of negative numbers, but for now we state they are not real numbers.

SIMPLIFYING HIGHER ROOTS

We can simplify higher roots in much the same way we simplified square roots, using the product rule of radicals.

PRODUCT RULE OF RADICALS

For any *nonnegative* real numbers a and b,

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{a\cdot b}$$

Often, we are not as familiar with perfect nth powers as we are with perfect squares. It is important to remember what index we are working with as we express higher roots in simplest radical form.

Example 2. Simplify.

$\sqrt[3]{54}$	We are working with a cube root, want perfect third powers. Test 2: $2^3=8$; 54 is not divisible by 8. Test 3: $3^3=27$; 54 is divisible by 27! Split radicand into factors
$=\sqrt[3]{27\cdot 2}$	Apply product rule
$=\sqrt[3]{27}\cdot\sqrt[3]{2}$	Take the cube root of 27
$=3\sqrt[3]{2}$	Our Answer

Just as with square roots, if we have a coefficient, then we multiply the new coefficients together.

Example 3. Simplify.

$3\sqrt[4]{48}$	We are working with a fourth root, want perfect fourth powers. Test 2: $2^4=16$; 48 is divisible by 16! Split radicand into factors
$=3\sqrt[4]{16\cdot 3}$	Apply product rule
$=3\sqrt[4]{16}\cdot\sqrt[4]{3}$	Take the fourth root of 16
$=3\cdot 2\sqrt[4]{3}$	Multiply coefficients
$=6\sqrt[4]{3}$	Our Answer

We can also take higher roots of variables. To simplify radical expressions involving variables, use the property below:

> **SIMPLIFYING $\sqrt[n]{a^n}$**
>
> For any *nonnegative* real number a,
>
> $\sqrt[n]{a^n} = a$

Note this property only holds if *a* is *nonnegative*. For this reason, we will assume that all variables involved in a radical expression are *nonnegative*.

As with square roots, when simplifying with variables, we will divide the variable's exponent by the index. The whole number part of the division is how many factors of that variable will come out of the n^th root. Any remainder is how many factors of the variable are left behind in the radicand. This process is shown in the following examples.

Example 4. Simplify.

$\sqrt[5]{x^{25}y^{17}z^3}$	Divide each exponent by 5: whole number outside, remainder inside
$= x^5y^3\sqrt[5]{y^2z^3}$	Our Answer

In the previous example, for the variable x, we divided $\frac{25}{5} = 5R0$, so x^5 came out and no xs remained inside. For y, we divided $\frac{17}{5} = 3R2$, so y^3 came out, and y^2 remained inside. For z, when we divided $\frac{3}{5} = 0R3$, all three or z^3 remained inside.

Example 5. Simplify.

$2\sqrt[3]{40a^4b^8}$	40 is divisible by the perfect cube 8 Split radicand into factors Apply product rule
$= 2\cdot\sqrt[3]{8}\cdot\sqrt[3]{5}\cdot\sqrt[3]{a^4}\cdot\sqrt[3]{b^8}$	Simplify roots; divide exponents by 3, remainders are left inside
$= 2\cdot 2ab^2\sqrt[3]{5ab^2}$	Multiply coefficients
$= 4ab^2\sqrt[3]{5ab^2}$	Our Answer

Practice Exercises
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Simplify.

1) $\sqrt[3]{64}$
2) $\sqrt[3]{-27}$
3) $\sqrt[4]{16}$
4) $\sqrt[4]{-16}$
5) $\sqrt[5]{-1}$
6) $\sqrt[8]{-1}$
7) $\sqrt[3]{625}$
8) $\sqrt[3]{375}$
9) $\sqrt[3]{750}$
10) $\sqrt[3]{250}$
11) $\sqrt[3]{24}$
12) $-4\sqrt[4]{96}$
13) $3\sqrt[4]{48}$
14) $-\sqrt[4]{112}$
15) $5\sqrt[4]{243}$
16) $\sqrt[4]{648a^2}$
17) $\sqrt[4]{64n^3}$
18) $\sqrt[5]{224n^3}$
19) $\sqrt[5]{-96x^3}$
20) $\sqrt[5]{224p^5}$
21) $\sqrt[6]{256x^6}$
22) $-8\sqrt[7]{384b^8}$
23) $-2\sqrt[3]{-48v^7}$
24) $-7\sqrt[3]{320n^6}$
25) $-\sqrt[3]{512n^6}$
26) $\sqrt[3]{-135x^5y^3}$
27) $\sqrt[3]{64u^5v^3}$
28) $\sqrt[3]{-32x^4y^4}$
29) $\sqrt[3]{1000a^4b^5}$
30) $\sqrt[3]{256x^4y^6}$
31) $7\sqrt[3]{-81x^3y^7}$
32) $-4\sqrt[3]{56x^2y^8}$
33) $8\sqrt[3]{-750xy}$
34) $-3\sqrt[3]{192ab^2}$
35) $3\sqrt[3]{135xy^3}$
36) $6\sqrt[3]{-54m^8n^3p^7}$
37) $-8\sqrt[4]{80m^4p^7q^4}$
38) $-2\sqrt[4]{405a^5b^8c}$
39) $7\sqrt[4]{128h^6j^8k^8}$
40) $5\sqrt[4]{324x^7yz^7}$

ANSWERS to Practice Exercises
Section 3.2: Higher Roots

1) 4
2) -3
3) 2
4) not a real number
5) -1
6) not a real number
7) $5\sqrt[3]{5}$
8) $5\sqrt[3]{3}$
9) $5\sqrt[3]{6}$
10) $5\sqrt[3]{2}$
11) $2\sqrt[3]{3}$
12) $-8\sqrt[4]{6}$
13) $6\sqrt[4]{3}$
14) $-2\sqrt[4]{7}$
15) $15\sqrt[4]{3}$
16) $3\sqrt[4]{8a^2}$
17) $2\sqrt[4]{4n^3}$
18) $2\sqrt[5]{7n^3}$
19) $-2\sqrt[5]{3x^3}$
20) $2p\sqrt[5]{7}$
21) $2x\sqrt[6]{4}$
22) $-16b\sqrt[7]{3b}$
23) $4v^2\sqrt[3]{6v}$
24) $-28n^2\sqrt[3]{5}$
25) $-8n^2$
26) $-3xy\sqrt[3]{5x^2}$
27) $4uv\sqrt[3]{u^2}$
28) $-2xy\sqrt[3]{4xy}$
29) $10ab\sqrt[3]{ab^2}$
30) $4xy^2\sqrt[3]{4x}$
31) $-21xy^2\sqrt[3]{3y}$
32) $-8y^2\sqrt[3]{7x^2y^2}$
33) $-40\sqrt[3]{6xy}$
34) $-12\sqrt[3]{3ab^2}$
35) $9y\sqrt[3]{5x}$
36) $-18m^2np^2\sqrt[3]{2m^2p}$
37) $-16mpq\sqrt[4]{5p^3}$
38) $-6ab^2\sqrt[4]{5ac}$
39) $14hj^2k^2\sqrt[4]{8h^2}$
40) $15xz\sqrt[4]{4x^3yz^3}$