

Section 3.6: Rational Exponents

**Objectives: Convert between radical notation and exponential notation.
Simplify expressions with rational exponents using the properties of exponents.
Multiply and divide radical expressions with different indices.**

We define rational exponents as follows:

DEFINITION OF RATIONAL EXPONENTS:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

The denominator of a rational exponent is the same as the index of our radical while the numerator serves as an exponent.

Either form of the definition can be used but we typically use the first form as it will involve smaller numbers.

Notice when the numerator of the exponent is 1, the special case of n^{th} roots follows from the definition:

$$a^{\frac{1}{n}} = (\sqrt[n]{a})^1 = \sqrt[n]{a}$$

CONVERTING BETWEEN EXPONENTIAL AND RADICAL NOTATION

We can use this definition to change any radical expression into an exponential expression.

Example 1. Rewrite with rational exponents.

$(\sqrt[5]{x})^3 = x^{\frac{3}{5}}$	$(\sqrt[6]{3x})^5 = (3x)^{\frac{5}{6}}$
$\frac{1}{(\sqrt[7]{a})^3} = a^{-\frac{3}{7}}$	$\frac{1}{(\sqrt[3]{xy})^2} = (xy)^{-\frac{2}{3}}$

Index is denominator, exponent is numerator

Negative exponents from reciprocals

We can also change any rational exponent into a radical expression by using the denominator as the index.

Example 2. Rewrite using radical notation.

$a^{\frac{5}{3}} = (\sqrt[3]{a})^5$	$(2mn)^{\frac{2}{7}} = (\sqrt[7]{2mn})^2$	Exponent is numerator; index is denominator
$x^{-\frac{4}{5}} = \frac{1}{(\sqrt[5]{x})^4}$	$(xy)^{-\frac{2}{9}} = \frac{1}{(\sqrt[9]{xy})^2}$	Negative exponent means reciprocals

The ability to change between exponential expressions and radical expressions allows us to evaluate expressions we had no means of evaluating previously.

Example 3. Use radical notation to rewrite and evaluate.

$$\begin{aligned}
 16^{\frac{3}{2}} & \quad \text{Change to radical format;} \\
 & \quad \text{numerator is exponent, denominator is index} \\
 = (\sqrt{16})^3 & \quad \text{Evaluate radical} \\
 = (4)^3 & \quad \text{Evaluate exponent} \\
 = 64 & \quad \text{Our Answer}
 \end{aligned}$$

Example 4. Use radical notation to rewrite and evaluate.

$$\begin{aligned}
 27^{-\frac{4}{3}} & \quad \text{Negative exponent is reciprocal} \\
 = \frac{1}{27^{\frac{4}{3}}} & \quad \text{Change to radical format;} \\
 & \quad \text{numerator is exponent, denominator is index} \\
 = \frac{1}{(\sqrt[3]{27})^4} & \quad \text{Evaluate radical} \\
 = \frac{1}{(3)^4} & \quad \text{Evaluate exponent} \\
 = \frac{1}{81} & \quad \text{Our Answer}
 \end{aligned}$$

SIMPLIFY EXPRESSIONS WITH RATIONAL EXPONENTS

The largest advantage of being able to change a radical expression into an exponential expression is we are now allowed to use all our exponent properties to simplify. The following table reviews all of our exponent properties.

PROPERTIES OF EXPONENTS		
$a^m a^n = a^{m+n}$	$(ab)^m = a^m b^m$	$a^{-m} = \frac{1}{a^m}$
$\frac{a^m}{a^n} = a^{m-n}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\frac{1}{a^{-m}} = a^m$
$(a^m)^n = a^{mn}$	$a^0 = 1$	$\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$

When adding and subtracting with fractions we need to have a common denominator. When multiplying we only need to multiply the numerators together and denominators together. The following examples show several different problems, using different properties to simplify the rational exponents.

Example 5. Simplify.

$$\begin{aligned}
 & a^{\frac{2}{3}} b^{\frac{1}{2}} a^{\frac{1}{6}} b^{\frac{1}{5}} && \text{Need common denominator for } a \text{ s (6) and for } b \text{ s (10)} \\
 & = a^{\frac{4}{6}} b^{\frac{5}{10}} a^{\frac{1}{6}} b^{\frac{2}{10}} && \text{Add exponents on } a \text{ s and } b \text{ s} \\
 & = a^{\frac{5}{6}} b^{\frac{7}{10}} && \text{Our Answer}
 \end{aligned}$$

Example 6. Simplify.

$$\begin{aligned}
 & \left(x^{\frac{1}{2}} y^{\frac{3}{5}}\right)^{\frac{3}{4}} && \text{Multiply each exponent by } \frac{3}{4}; \\
 & && \text{reduce fractions} \\
 & = x^{\frac{1}{4}} y^{\frac{3}{10}} && \text{Our Answer}
 \end{aligned}$$

Example 7. Simplify.

$$\frac{x^2 y^{\frac{2}{3}}}{x^{\frac{7}{2}} y^0}$$

Need common denominator for x s (2) to subtract exponents

$$= \frac{x^{\frac{4}{2}} y^{\frac{2}{3}}}{x^{\frac{7}{2}} y^0}$$

Subtract exponents on x in denominator, $y^0 = 1$

$$= x^{-\frac{3}{2}} y^{\frac{2}{3}}$$

Negative exponent moves down to denominator

$$= \frac{y^{\frac{2}{3}}}{x^{\frac{3}{2}}}$$

Our Answer

MULTIPLY AND DIVIDE RADICAL EXPRESSIONS WITH DIFFERENT INDICES

We will use rational exponents to multiply or divide radical expressions having different indices. We will convert each radical expression to its equivalent exponential expression. Then, we will apply the appropriate exponent property. For our answer, we will convert the exponential expression to its equivalent radical expression. Our answer will then be written as a single radical expression.

Example 8. Multiply, writing the expression using a single radical.

$$\sqrt[5]{x} \cdot \sqrt{x}$$

Rewrite radical expressions using rational exponents

$$= x^{\frac{1}{5}} \cdot x^{\frac{1}{2}}$$

Need common denominator of 10 to add exponents

$$= x^{\frac{2}{10}} \cdot x^{\frac{5}{10}}$$

Add exponents

$$= x^{\frac{7}{10}}$$

Rewrite as a radical expression

$$= \sqrt[10]{x^7}$$

Our Answer

Example 9. Divide, writing the expression using a single radical.

$$\frac{\sqrt[3]{y^2}}{\sqrt[5]{y^2}}$$

Rewrite radical expressions using rational exponents

$$= \frac{y^{\frac{2}{3}}}{y^{\frac{2}{5}}}$$

Need common denominator of 15 to subtract exponents

$$= \frac{y^{\frac{10}{15}}}{y^{\frac{6}{15}}}$$

Subtract exponents

$$= y^{\frac{4}{15}}$$

Rewrite as a radical expression

$$= \sqrt[15]{y^4}$$

Our Answer

It is important to remember that as we simplify with rational exponents, we are using the exact same properties we used when simplifying integer exponents. The only difference is we need to follow our rules for fractions as well. It may be worth reviewing your notes on exponent properties to be sure you are comfortable with using the properties.

Practice Exercises

Section 3.6: Rational Exponents

Write each expression in radical form.

1) $m^{\frac{3}{5}}$

3) $(7x)^{\frac{3}{2}}$

2) $(10r)^{-\frac{3}{4}}$

4) $(6b)^{-\frac{3}{4}}$

Write each expression in exponential form.

5) $\frac{1}{(\sqrt{6x})^3}$

7) $\frac{1}{(\sqrt[4]{n})^7}$

6) \sqrt{v}

8) $\sqrt{5a}$

Evaluate.

9) $8^{\frac{2}{3}}$

13) $27^{-\frac{1}{3}}$

10) $16^{\frac{1}{4}}$

14) $32^{\frac{3}{5}}$

11) $4^{\frac{3}{2}}$

15) $81^{-\frac{3}{4}}$

12) $100^{-\frac{3}{2}}$

16) $25^{\frac{3}{2}}$

Simplify. Your answer should contain only positive exponents.

17) $x^{\frac{1}{3}}y \cdot xy^{\frac{3}{2}}$

20) $(x^{\frac{5}{3}}y^{-2})^0$

18) $4v^{\frac{2}{3}} \cdot v^{-1}$

21) $(x^0y^{\frac{1}{3}})^{\frac{3}{2}}x^0$

19) $(a^{\frac{1}{2}}b^{\frac{1}{2}})^{-1}$

22) $u^{-\frac{5}{4}}v^2 \cdot (u^{\frac{3}{2}})^{-\frac{3}{2}}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 3.6 (continued)

Simplify. Your answer should contain only positive exponents.

$$23) \frac{a^{\frac{3}{4}}b^{-1} \cdot b^{\frac{7}{4}}}{3b^{-1}}$$

$$28) \frac{(y^{-\frac{1}{2}})^{\frac{3}{2}}}{x^{\frac{3}{2}}y^{\frac{1}{2}}}$$

$$24) \frac{2x^{-2}y^{\frac{5}{3}}}{x^{-\frac{5}{4}}y^{-\frac{5}{3}} \cdot xy^{\frac{1}{2}}}$$

$$29) \frac{(m^2n^{\frac{1}{2}})^0}{n^{\frac{3}{4}}}$$

$$25) \frac{3y^{-\frac{5}{4}}}{y^{-1} \cdot 2y^{-\frac{1}{3}}}$$

$$30) \frac{y^0}{(x^{\frac{3}{4}}y^{-1})^{\frac{1}{3}}}$$

$$26) \frac{ab^{\frac{1}{3}} \cdot 2b^{-\frac{5}{4}}}{4a^{-\frac{1}{2}}b^{\frac{2}{3}}}$$

$$31) \frac{(x^{-\frac{4}{3}}y^{-\frac{1}{3}} \cdot y)^{-1}}{x^{\frac{1}{3}}y^{-2}}$$

$$27) \left(\frac{m^{\frac{3}{2}}n^{-2}}{(mn^{\frac{4}{3}})^{-1}} \right)^{\frac{7}{4}}$$

$$32) \frac{(x^{\frac{1}{2}}y^0)^{-\frac{4}{3}}}{y^4 \cdot x^{-2}y^{-\frac{2}{3}}}$$

Perform the indicated operation, writing the expression using a single radical.

$$33) \sqrt{x} \cdot \sqrt[4]{x}$$

$$34) \frac{\sqrt[5]{x^2}}{\sqrt[6]{x}}$$

ANSWERS to Practice Exercises
Section 3.6: Rational Exponents

1) $(\sqrt[5]{m})^3$

3) $(\sqrt{7x})^3$

2) $\frac{1}{(\sqrt[4]{10r})^3}$

4) $\frac{1}{(\sqrt[4]{6b})^3}$

5) $(6x)^{-\frac{3}{2}}$

7) $n^{-\frac{7}{4}}$

6) $v^{\frac{1}{2}}$

8) $(5a)^{\frac{1}{2}}$

9) 4

13) $\frac{1}{3}$

10) 2

14) 8

11) 8

12) $\frac{1}{1000}$

15) $\frac{1}{27}$

16) 125

17) $x^{\frac{4}{3}}y^{\frac{5}{2}}$

20) 1

18) $\frac{4}{v^{\frac{1}{3}}}$

21) $y^{\frac{1}{2}}$

19) $\frac{1}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

22) $\frac{v^2}{u^{\frac{7}{2}}}$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 3.6 (continued)

23) $\frac{a^{\frac{3}{2}}b^{\frac{7}{4}}}{3}$

28) $\frac{1}{x^{\frac{3}{2}}y^{\frac{5}{4}}}$

24) $\frac{2y^{\frac{17}{6}}}{x^{\frac{7}{4}}}$

29) $\frac{1}{n^{\frac{3}{4}}}$

25) $\frac{3y^{\frac{1}{12}}}{2}$

30) $\frac{y^{\frac{1}{3}}}{x^{\frac{1}{4}}}$

26) $\frac{a^{\frac{3}{2}}}{2b^{\frac{19}{12}}}$

31) $xy^{\frac{4}{3}}$

27) $\frac{m^{\frac{35}{8}}}{n^{\frac{7}{6}}}$

32) $\frac{x^{\frac{4}{3}}}{y^{\frac{10}{3}}}$

33) $\sqrt[4]{x^3}$

34) $\sqrt[30]{x^7}$

