# Section 3.8: Complex Numbers

**Objective: Add, subtract, multiply, divide, and simplify expressions using complex numbers.**

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, mathematicians create new ways for dealing with the problem. Throughout history this has been the case with the need for a number that is nothing $(0)$, smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for the square roots of negative numbers. To work with the square root of negative numbers and to solve equations such as $x^2 = -1$; mathematicians have created a new number system called the complex numbers. First, we define the imaginary unit:

**DEFINITION OF THE IMAGINARY UNIT *i***

$$i = \sqrt{-1} \quad \text{where} \quad i^2 = -1$$

With this definition, the square root of a negative number can now be expressed as a multiple of $i$. We will use the product rule of radicals and simplify the negative as a factor of negative one. This process is shown in the following examples.

**Example 1.** Write in terms of $i$.

| | |
|---|---|
| $\sqrt{-16}$ | Consider the negative as a factor of $-1$ |
| $= \sqrt{-1 \cdot 16}$ | Take each root, square root of $-1$ is $i$ |
| $= 4i$ | Our Answer |

**Example 2.** Write in terms of $i$.

| | |
|---|---|
| $\sqrt{-24}$ | Find perfect square factors, including $-1$ |
| $= \sqrt{-1 \cdot 4 \cdot 6}$ | Square root of $-1$ is $i$, square root of $4$ is $2$ |
| $= 2i\sqrt{6}$ | Our Answer |

Then, mathematicians created a new number system called the set of complex numbers. A **complex number** is one that contains both a real and imaginary part.

**DEFINITION OF COMPLEX NUMBERS**

$$a+bi$$

where $a$ and $b$ are real numbers

We call $a$ the real part and $b$ the imaginary part. Examples of complex numbers include $2+5i, -3+i\sqrt{5}, -6i$ because $-6i = 0-6i$ ; and $3$ because $3 = 3+0i$ .

## ADDING AND SUBTRACTING COMPLEX NUMBERS

The operations of addition, subtraction, and multiplication of complex numbers are performed very similarly to how they are done with polynomials. A new technique will be needed for division though.

When adding and subtracting complex numbers, we combine like terms by adding or subtracting the real parts, adding or subtracting the imaginary parts, and expressing the answer in the form of a complex number $a+bi$.

**Example 3.** Add.

| | |
|---|---|
| $(2+5i)+(4-7i)$ | Combine real parts, $2+4$ imaginary parts $5i-7i$ |
| $=6-2i$ | Our Answer |

It is important to notice what operation we are doing. Students often see the parentheses and think that means FOIL. We only use FOIL to multiply. This problem is an addition problem, so we simply add the real and imaginary parts.

For subtraction of complex numbers, the idea is the same, but we need to remember to first distribute the negative onto all the terms in the parentheses.

**Example 4.** Subtract.

| | |
|---|---|
| $(4-8i)-(3-5i)$ | Distribute the negative |
| $=4-8i-3+5i$ | Combine real parts, $4-3$ imaginary parts $-8i+5i$ |
| $=1-3i$ | Our Answer |

Addition and subtraction can appear together in one problem.

**Example 5.** Perform the indicated operations.

| | |
|---|---|
| $(5i)-(3+8i)+(-4+7i)$ | Distribute the negative |
| $=5i-3-8i-4+7i$ | Combine real parts, $-3-4$ imaginary parts $5i-8i+7i$ |
| $=-7+4i$ | Our Answer |

## MULTIPLYING COMPLEX NUMBERS

Multiplying complex numbers is the same as multiplying polynomials, but we replace $i^2$ with $-1$.

**Example 6.** Multiply.

| | |
|---|---|
| $(3i)(7i)$ | Multiply |
| $=21i^2$ | Replace $i^2$ with $-1$ |
| $=21(-1)$ | Multiply |
| $=-21$ | Our Answer |

When multiplying complex radicals, it is important that we first rewrite as multiples of $i$.

**Example 7.** Multiply.

| | |
|---|---|
| $\sqrt{-6}\sqrt{-3}$ | Simplify each root using $i=\sqrt{-1}$ |
| $=(i\sqrt{6})(i\sqrt{3})$ | Multiply |
| $=i^2\sqrt{18}$ | Replace $i^2$ with $-1$ |
| $=-\sqrt{18}$ | Simplify the radical |
| $=-\sqrt{9\cdot 2}$ | Take square root of $9$ |
| $-3\sqrt{2}$ | Our Answer |

**Example 8.** Multiply.

| | |
|---|---|
| $5i(3i-7)$ | Distribute |
| $=15i^2-35i$ | Replace $i^2$ with $-1$ |
| $=15(-1)-35i$ | Multiply |
| $=-15-35i$ | Our Answer |

**Example 9.** Multiply.

| | |
|---|---|
| $(2-4i)(3+5i)$ | FOIL |
| $=6+10i-12i-20i^2$ | Replace $i^2$ with $-1$ |
| $=6+10i-12i-20(-1)$ | Multiply |
| $=6+10i-12i+20$ | Combine real parts, $6+20$ imaginary parts $10i-12i$ |
| $=26-2i$ | Our Answer |

Remember when squaring a binomial, we write as a product of two same binomials and then FOIL.

**Example 10.** Multiply.

| | |
|---|---|
| $(4-5i)^2$ | Write as a product of two same complex numbers |
| $=(4-5i)(4-5i)$ | FOIL |
| $=16-20i-20i+25i^2$ | Replace $i^2$ with $-1$ |
| $=16-20i-20i+25(-1)$ | Multiply |
| $=16-20i-20i-25$ | Combine real parts, $16-25$ imaginary parts $-20i-20i$ |
| $=-9-40i$ | Our Answer |

**Example 11.** Multiply.

| | |
|---|---|
| $(2+3i)(2-3i)$ | FOIL |
| $=4-6i+6i-9i^2$ | Replace $i^2$ with $-1$ |
| $=4-6i+6i-9(-1)$ | Multiply |
| $=4-6i+6i+9$ | Combine real parts, $4+9$ imaginary parts $-6i+6i$ |
| $=13$ | Our Answer |

Notice how the product of the two complex numbers above resulted in a real number.

The complex numbers $a+bi$ and $a-bi$ are called **complex conjugates** of each other. Notice that $(a+bi)\,(a-bi)=a^2+b^2$. When we multiply complex conjugates, the result is always a real number.

## DIVIDING COMPLEX NUMBERS

Dividing complex numbers also has one thing we need to be careful of. If $i$ is $\sqrt{-1}$, and it is in the denominator of a fraction, then we have a radical in the denominator! This means we will want to rationalize our denominator so there are no $i$ s. This is done by multiplying numerator and denominator by the conjugate of the denominator.

**Example 12.** Divide.

| | |
|---|---|
| $\frac{2-6i}{4+8i}$ | Multiply by conjugate of denominator, $4-8i$ |
| $=\frac{(2-6i)}{(4+8i)}\cdot\frac{(4-8i)}{(4-8i)}$ | FOIL in numerator, denominator is difference of squares |
| $=\frac{8-16i-24i+48i^2}{16-64(-1)}$ | Replace $i^2$ with $-1$ |
| $=\frac{8-16i-24i+48(-1)}{16-64(-1)}$ | Multiply |
| $=\frac{8-16i-24i-48}{16+64}$ | Combine real and imaginary parts |
| $=\frac{-40-40i}{80}$ | Reduce, dividing each term by 80 |
| $=-\frac{40}{80}-\frac{40i}{80}$ | Reduce and write in the form of a complex number |
| $=-\frac{1}{2}-\frac{1}{2}i$ | Our Answer |

**Example 13.** Divide.

| | |
|---|---|
| $=\frac{7+3i}{-5i}$ | Multiply by conjugate of denominator, $5i$ |
| $=\frac{(7+3i)}{-5i}\cdot\frac{5i}{5i}$ | Distribute $5i$ in numerator |
| $=\frac{35i+15i^2}{-25i^2}$ | Replace $i^2$ with $-1$ |

$= \frac{35i + 15(-1)}{-25(-1)}$ Multiply

$= \frac{35i - 15}{25}$ Reduce, dividing each term by 25

$= -\frac{15}{25} + \frac{35}{25}i$ Reduce and write in the form of a complex number

$= -\frac{3}{5} + \frac{7}{5}i$ Our Answer

# Practice Exercises
## Section 3.8: Complex Numbers

**Write in terms of $i$.**

1) $\sqrt{-81}$
2) $\sqrt{-45}$

**Multiply.**

3) $\sqrt{-4}\cdot\sqrt{-9}$
4) $\sqrt{-10}\cdot\sqrt{-2}$
5) $\sqrt{-12}\cdot\sqrt{-2}$
6) $\sqrt{-3}\cdot\sqrt{27}$

**Perform the indicated operation, writing the answer in the form of a complex number $a+bi$.**

7) $3-(-8+4i)$
8) $(-8i)-(7i)-(5-3i)$
9) $(7i)-(3-2i)$
10) $(-4-i)+(1-5i)$
11) $(-6i)-(3+7i)$
12) $(5-4i)+(8-4i)$
13) $(3-3i)+(-7-8i)$
14) $(i)-(2+3i)-6$
15) $(3i)(-8i)$
16) $(16i)(-2i)$
17) $(6i)(-9i)$
18) $(-7i)^2$
19) $(-5i)(-10i)$
20) $(-7-4i)(-8+6i)$
21) $(6+5i)^2$
22) $(8-6i)(-4+2i)$
23) $(-4+5i)(2-7i)$
24) $(-2+i)(3-5i)$
25) $(1+5i)(2+i)$
26) $\frac{9}{i}$

*The Practice Exercises are continued on the next page.*

*Practice Exercises: Section 3.8 (continued)*

**Perform the indicated operation, writing the answer in the form of a complex number $a+bi$.**

27) $\frac{5}{6i}$

28) $\frac{-3+2i}{-3i}$

29) $\frac{-3-6i}{4i}$

30) $\frac{-4+2i}{3i}$

31) $\frac{10-i}{-i}$

32) $\frac{4i}{-10+i}$

33) $\frac{8}{7-6i}$

34) $\frac{9i}{1-5i}$

35) $\frac{7}{10-7i}$

36) $\frac{4}{4+6i}$

37) $\frac{5-3i}{3+2i}$

38) $\frac{1+7i}{1+i}$

39) $\frac{6-i}{4-3i}$

40) $\frac{3+8i}{2-5i}$

# ANSWERS to Practice Exercises
## Section 3.8: Complex Numbers

1) $9i$

2) $3i\sqrt{5}$

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3) $-6$

4) $-2\sqrt{5}$

5) $-2\sqrt{6}$

6) $9i$

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7) $11-4i$

8) $-5-12i$

9) $-3+9i$

10) $-3-6i$

11) $-3-13i$

12) $13-8i$

13) $-4-11i$

14) $-8-2i$

15) $24$

16) $32$

17) $54$

18) $-49$

19) $-50$

20) $80-10i$

21) $11+60i$

22) $-20+40i$

23) $27+38i$

24) $-1+13i$

25) $-3+11i$

26) $-9i$

*The Answers to Practice Exercises are continued on the next page.*

*ANSWERS to Practice Exercises: Section 3.8 (continued)*

27) $-\frac{5}{6}i$

28) $-\frac{2}{3}-i$

29) $-\frac{3}{2}+\frac{3}{4}i$

30) $\frac{2}{3}+\frac{4}{3}i$

31) $1+10i$

32) $\frac{4}{101}-\frac{40}{101}i$

33) $\frac{56}{85}+\frac{48}{85}i$

34) $-\frac{45}{26}+\frac{9}{26}i$

35) $\frac{70}{149}+\frac{49}{149}i$

36) $\frac{4}{13}-\frac{6}{13}i$

37) $\frac{9}{13}-\frac{19}{13}i$

38) $4+3i$

39) $\frac{27}{25}+\frac{14}{25}i$

40) $-\frac{34}{29}+\frac{31}{29}i$