# Section 4.3: Quadratic Formula

## Objective: Solve quadratic equations using the quadratic formula.

In this section, we will develop a formula to solve any quadratic equation $ax^2+bx+c=0$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$. Solve for this general equation for $x$ by completing the square:

$$ax^2+bx+c=0$$

Separate the constant term from variable terms

$$\underline{-c=-c}$$

Subtract $c$ from both sides

$$\frac{ax^2}{a}+\frac{bx}{a} = \frac{-c}{a}$$

Divide each term by $a$

$$x^2+\frac{b}{a}x = \frac{-c}{a}$$

Find the value that completes the square:

$$\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2=\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$$

Add that value to both sides of the equation

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

Subtract fractions on the right side of the equation using a common denominator of $4a^2$:

$$\frac{b^2}{4a^2}-\frac{c}{a}\left(\frac{4a}{4a}\right)=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}=\frac{b^2-4ac}{4a^2}$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

Factor the perfect square trinomial

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Solve by applying the square root property

$$\sqrt{\left(x+\frac{b}{2}\right)^2}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Simplify radicals

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Our Solutions

# QUADRATIC FORMULA

This result is a very important one to us because we can use this formula to solve any quadratic equation. Once we identify the values of *a; b;* and *c* in the quadratic equation, we can substitute those values into $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and we get our solutions. This formula is known as the **quadratic formula**.

**QUADRATIC FORMULA:**

The solutions to the quadratic equation $ax^2 + bx + c = 0$ for $a \neq 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

# SOLVING QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA

We can use the quadratic formula to solve any quadratic equation. This method is demonstrated in the following examples.

**Example 1**. Solve the equation using the quadratic formula.

$x^2 + 3x + 2 = 0$ — $a = 1$, $b = 3$, $c = 2$; use quadratic formula

$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(2)}}{2(1)}$ — Evaluate the exponent and multiply

$x = \frac{-3 \pm \sqrt{9 - 8}}{2}$ — Evaluate the subtraction under radical sign

$x = \frac{-3 \pm \sqrt{1}}{2}$ — Evaluate the root

$x = \frac{-3 \pm 1}{2}$ — Evaluate $+$ and $-$ to get the two answers

$x = \frac{-3+1}{2}$ or $x = \frac{-3-1}{2}$

$x = \frac{-2}{2}$ or $x = \frac{-4}{2}$ — Simplify the fractions, if possible

$x = -1$ or $-2$ — Our Solutions

As we are solving a quadratic equation using the quadratic formula, it is important to remember that the equation must first be set equal to zero.

**Example 2**. Solve the equation using the quadratic formula.

| | |
|---|---|
| $25x^2 = 30x + 11$ | First set the equation equal to zero |
| $-30x - 11 = -30x - 11$ | Subtract $30x$ and 11 from both sides of the equation |
| $25x^2 - 30x - 11 = 0$ | $a = 25$, $b = -30$, and $c = -11$; use quadratic formula |
| $x = \dfrac{-(-30) \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)}$ | Evaluate the exponent and multiply |
| $x = \dfrac{30 \pm \sqrt{900 + 1100}}{50}$ | Evaluate the addition under radical sign |
| $x = \dfrac{30 \pm \sqrt{2000}}{50}$ | Simplify the root |
| $x = \dfrac{30 \pm 20\sqrt{5}}{50}$ | Factor numerator and denominator |
| $x = \dfrac{10\left(3 \pm 2\sqrt{5}\right)}{10 \cdot 5}$ | Divide out common factor of 10 |
| $x = \dfrac{3 \pm 2\sqrt{5}}{5}$ | Our Solutions |

**Example 3**. Solve the equation using the quadratic formula.

| | |
|---|---|
| $3x^2 + 4x + 8 = 2x^2 + 6x - 5$ | First set the equation equal to zero |
| $-2x^2 - 6x + 5 = -2x^2 - 6x + 5$ | Subtract $2x^2$ and $6x$, and add 5 |
| $x^2 - 2x + 13 = 0$ | $a = 1$, $b = -2$, and $c = 13$; use quadratic formula |
| $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)}$ | Evaluate the exponent and multiply |
| $x = \dfrac{2 \pm \sqrt{4 - 52}}{2}$ | Evaluate the subtraction under radical sign |

$$x=\frac{2\pm\sqrt{-48}}{2}$$ Simplify the root

$$x=\frac{2\pm 4i\sqrt{3}}{2}$$ Factor numerator

$$x=\frac{2\left(1\pm 2i\sqrt{3}\right)}{2\cdot 1}$$ Divide out common factor of 2

$$x=1\pm 2i\sqrt{3}$$ Our Solutions

Notice this equation has two imaginary solutions and they are complex conjugates.

When we solve quadratic equations, we don't necessarily get two unique solutions. We can end up with only one real number solution if the square root simplifies to zero.

**Example 4**. Solve the equation using the quadratic formula.

$$4x^2-12x+9=0$$ $a=4$, $b=-12$, $c=9$; use quadratic formula

$$x=\frac{-(-12)\pm\sqrt{(-12)^2-4(4)(9)}}{2(4)}$$ Evaluate the exponent and multiply

$$x=\frac{12\pm\sqrt{144-144}}{8}$$ Evaluate the subtraction under radical sign

$$x=\frac{12\pm\sqrt{0}}{8}$$ Simplify the root

$$x=\frac{12\pm 0}{8}$$ Evaluate $+$ and $-$ to get the two answers;

They are identical values; so, only one instance needs to be considered

$$x=\frac{12}{8}$$ Reduce fraction

$$x=\frac{3}{2}$$ Our Solution

When solving a quadratic equation, if the term with $x$ or the constant term is missing, we can still solve the equation using the quadratic formula. We simply use zero for the coefficient of the missing term. If the term with $x$ is missing, we have $b=0$ and if the constant term is missing, we have $c=0$. Note if $a=0$, the term with $x^2$ is missing, meaning the equation is a linear equation, not a quadratic equation.

**Example 5**. Solve the equation using the quadratic formula.

$$3x^2+7=0$$ $a=3$, $b=0$ (missing term), $c=7$; use quadratic formula

$$x=\frac{-(0)\pm\sqrt{(0)^2-4(3)(7)}}{2(3)}$$ Evaluate the exponent and multiply

$$x=\pm\frac{\sqrt{-84}}{6}$$ Simplify the root

$$x=\pm\frac{2i\sqrt{21}}{6}$$ Reduce the fraction; divide $2i$ and $6$ by $2$

$$x=\pm\frac{i\sqrt{21}}{3}$$ Our Solutions

## SELECTING A METHOD FOR SOLVING A QUADRATIC EQUATION

We have covered four different methods that can be used to solve a quadratic equation: factoring, applying the square root property, completing the square, and using the quadratic formula. It is important to be familiar with all four methods as each has its own advantages when solving quadratic equations.

Some of the examples in this section could have been solved using a method other than the quadratic formula. In Example 1, we used the quadratic formula to solve the equation $x^2+3x+2=0$. We could have chosen to solve this equation factoring instead:

$$x^2+3x+2=0$$
$$(x+1)(x+2)=0$$
$$x+1=0 \qquad x+2=0$$
$$x=-1 \qquad x=-2$$

In Example 5, we could have chosen to solve $3x^2+7=0$ by applying the square root property since there is no $x$ term and we can isolate the squared term.

The following table walks you through a suggested process and an example of each method to decide which would be best to use when solving a quadratic equation.

| | |
|---|---|
| 1. If $ax^2+bx+c$ can be factored easily, solve by **factoring**: | $x^2-5x+6=0$<br>$(x-2)(x-3)=0$<br>$x=2$ or $x=3$ |
| 2. If the equation can be written with a squared term or expression on one side and a constant term on the other, solve by applying the **square root property**: | $x^2-7=0$<br>$x^2=7$<br>$x=\pm\sqrt{7}$ |
| 3. If $a=1$ and $b$ is even, solve by **completing the square**: | $x^2+2x=4$<br>$\left(\frac{1}{2}\cdot 2\right)^2=1^2=1$<br>$x^2+2x+1=4+1$<br>$(x+1)^2=5$<br>$x+1=\pm\sqrt{5}$<br>$x=-1\pm\sqrt{5}$ |
| 4. Otherwise, solve by the **quadratic formula**: | $x^2-3x+4=0$<br>$x=\frac{3\pm\sqrt{(-3)^2-4(1)(4)}}{2(1)}$<br>$x=\frac{3\pm i\sqrt{7}}{2}$ |

The above table offers a suggestion for deciding how to solve a quadratic equation. Remember that the methods of completing the square and the quadratic formula will always work to solve any quadratic equation. Solving a quadratic equation by factoring only works if the expression can be factored.

# Practice Exercises
## Section 4.3: Quadratic Formula

**Solve each equation using the quadratic formula.**

1) $x^2 - 4x + 3 = 0$
2) $m^2 + 4m - 48 = -3$
3) $4x^2 - 7 = -5x$
4) $3k^2 = -3k + 11$
5) $2x^2 + 4x = -4$
6) $5p^2 + 2p + 6 = 0$
7) $3r^2 - 2r - 1 = 0$
8) $2x^2 - 2x - 15 = 0$
9) $4n^2 - 36 = 0$
10) $3b^2 + 6 = 0$
11) $2x^2 + 3x + 8 = 0$
12) $3x = 6x^2 + 7$
13) $2x^2 = 8x + 2$
14) $-16t^2 + 32t + 48 = 0$
15) $7x^2 + 3x = 14$
16) $6n^2 - 1 = 0$
17) $2p^2 + 6p - 16 = 4$
18) $9m^2 - 16 = 0$
19) $3n^2 + 3n = -3$
20) $3t^2 - 3 = 8t$
21) $2x^2 = -7x + 49$
22) $-3r^2 + 4 = -6r$
23) $5x^2 = 7x + 7$
24) $6a^2 = -5a + 13$
25) $8n^2 = -3n - 8$
26) $6v^2 = 4 + 6v$
27) $2x^2 + 5x = -3$
28) $x^2 = 8$
29) $4a^2 - 64 = 0$
30) $2k^2 + 6k - 16 = 2k$
31) $4p^2 + 5p - 36 = 3p^2$
32) $12x^2 + x + 7 = 5x^2 + 5x$
33) $-5n^2 - 3n - 52 = 2 - 7n^2$
34) $7m^2 - 6m + 6 = -m$
35) $7r^2 - 12 = -3r$
36) $3x^2 - 3 = x^2$
37) $2n^2 - 9 = 4$
38) $6t^2 = t^2 + 7 - t$

# ANSWERS to Practice Exercises
## Section 4.3: Quadratic Formula

1) $1, 3$
2) $5, -9$
3) $\dfrac{-5+\sqrt{137}}{8}, \dfrac{-5-\sqrt{137}}{8}$
4) $\dfrac{-3+\sqrt{141}}{6}, \dfrac{-3-\sqrt{141}}{6}$
5) $-1+i, -1-i$
6) $\dfrac{-1+i\sqrt{29}}{5}, \dfrac{-1-i\sqrt{29}}{5}$
7) $1, -\dfrac{1}{3}$
8) $\dfrac{1+\sqrt{31}}{2}, \dfrac{1-\sqrt{31}}{2}$
9) $3, -3$
10) $i\sqrt{2}, -i\sqrt{2}$
11) $\dfrac{-3+i\sqrt{55}}{4}, \dfrac{-3-i\sqrt{55}}{4}$
12) $\dfrac{3+i\sqrt{159}}{12}, \dfrac{3-i\sqrt{159}}{12}$
13) $2+\sqrt{5}, 2-\sqrt{5}$
14) $3, -1$
15) $\dfrac{-3+\sqrt{401}}{14}, \dfrac{-3-\sqrt{401}}{14}$
16) $\dfrac{\sqrt{6}}{6}, -\dfrac{\sqrt{6}}{6}$
17) $2, -5$
18) $\dfrac{4}{3}, -\dfrac{4}{3}$
19) $\dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$
20) $3, -\dfrac{1}{3}$
21) $\dfrac{7}{2}, -7$
22) $\dfrac{3+\sqrt{21}}{3}, \dfrac{3-\sqrt{21}}{3}$
23) $\dfrac{7+3\sqrt{21}}{10}, \dfrac{7-3\sqrt{21}}{10}$
24) $\dfrac{-5+\sqrt{337}}{12}, \dfrac{-5-\sqrt{337}}{12}$
25) $\dfrac{-3+i\sqrt{247}}{16}, \dfrac{-3-i\sqrt{247}}{16}$
26) $\dfrac{3+\sqrt{33}}{6}, \dfrac{3-\sqrt{33}}{6}$
27) $-1, -\dfrac{3}{2}$
28) $2\sqrt{2}, -2\sqrt{2}$
29) $4, -4$
30) $2, -4$
31) $4, -9$
32) $\dfrac{2+3i\sqrt{5}}{7}, \dfrac{2-3i\sqrt{5}}{7}$
33) $6, -\dfrac{9}{2}$
34) $\dfrac{5+i\sqrt{143}}{14}, \dfrac{5-i\sqrt{143}}{14}$
35) $\dfrac{-3+\sqrt{345}}{14}, \dfrac{-3-\sqrt{345}}{14}$
36) $\dfrac{\sqrt{6}}{2}, -\dfrac{\sqrt{6}}{2}$
37) $\dfrac{\sqrt{26}}{2}, -\dfrac{\sqrt{26}}{2}$
38) $\dfrac{-1+\sqrt{141}}{10}, \dfrac{-1-\sqrt{141}}{10}$