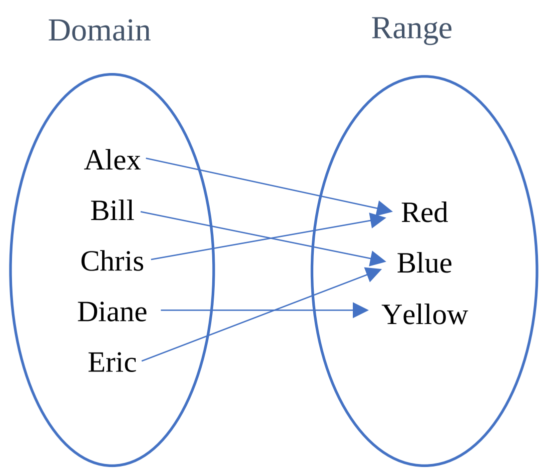
Section 5.1: Functions

Objective: Identify functions and use correct notation to evaluate functions at numerical and variable values.

A relationship is a matching of elements between two sets with the first set called the **domain** and the second set called the **range**. The following examples show several ways that relationships can be given.

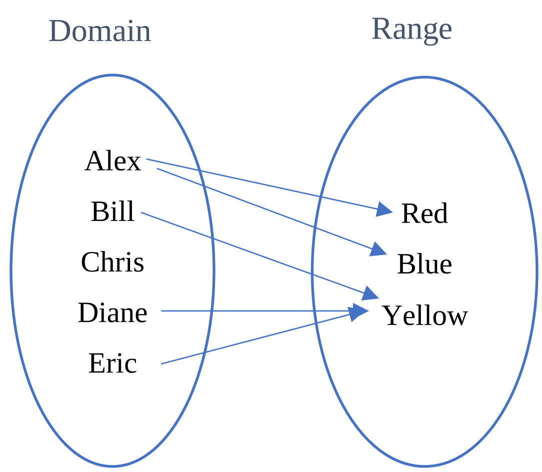
Example 1. Suppose that the domain is five people in a classroom, {Alex, Bill, Chris, Diane, Eric} and the range is the colors, {red, blue, yellow}. Suppose the relationship shows each person's favorite color from those three choices. This can be shown with a **mapping**.



Example 2. Another way to express a relationship is to give a set of ordered pairs. The set {(Alex, Red), (Bill, Blue), (Chris, Red), (Diane, Yellow), (Eric, Blue)} gives the exact same information as the mapping above. In a relationship, some values might be matched more than once. Note that in the above example that each person is matched with just one color but some colors are used more than once.

Example 3. Suppose the following mapping uses the same domain and range as the first two examples but now is showing the information about which colors each person is wearing.

a) Who is wearing Red? (Answer: Alex)
b) Which of those colors is Bill wearing? (Answer: Yellow)
c) Who is wearing Yellow? (Answer: Bill and Diane)



Notice that Alex is wearing both red and blue clothing. Chris is not wearing any of the colors listed in the range.

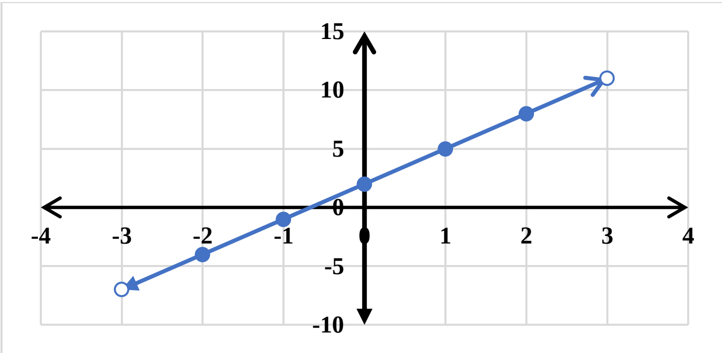
Another way to express a relationship is with an **equation**. Normally *y* denotes the range and *x* the domain.

Example 4. If we have the equation $y = 3x + 2$ then we have a relationship where the domain and range are both the set of all real numbers. Here we can find the matching *y* for any value of *x*. For example, if $x = 0$ then $y = 3(0) + 2 = 2$. This can be expressed as the ordered pair $(0, 2)$. Other ordered pairs that satisfy the equation include $(1, 5)$ and $(7, 23)$.

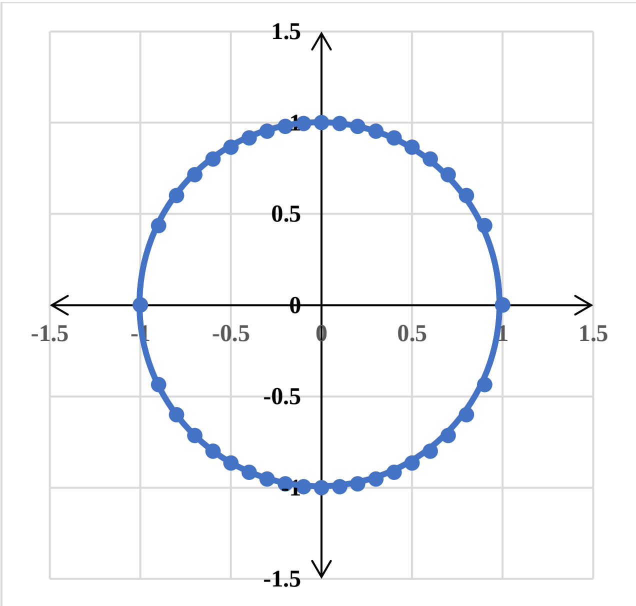
Example 5. If we have the equation $y^2 + x^2 = 1$ then again we have a relationship. Here note that for a given value of *x* that there might be more than one matching value of *y*. For example, if $x = 0$, then $y = -1$ or $y = 1$. Both make the equation true.

Another way to express a relationship is with a **graph**.

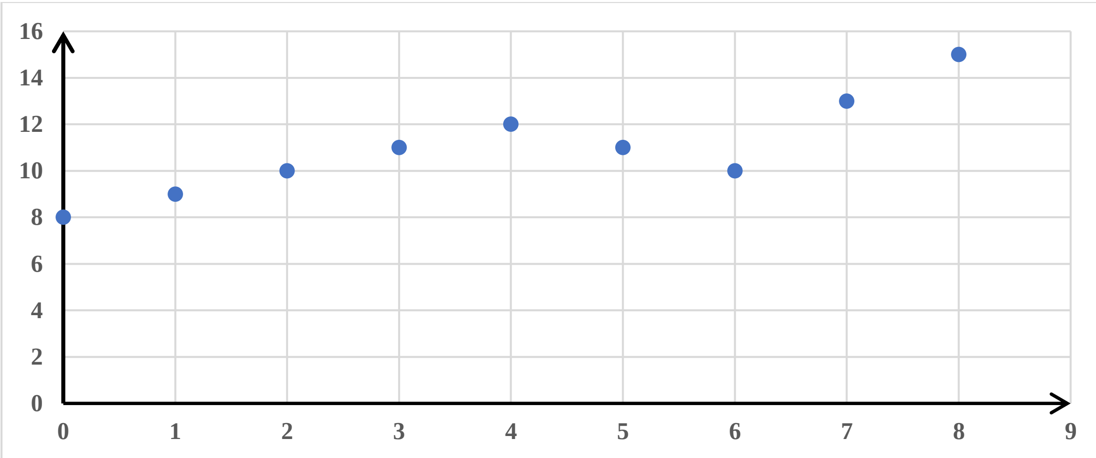
Example 6. This is the graph of the equation $y = 3x + 2$ from Example 4. The graph shows the combination of *x* and *y* values which make the equation true.



Example 7. This is the graph of the equation $y^2 + x^2 = 1$ from Example 5. The graph shows the combination of *x* and *y* values which make the equation true.



Example 8. A graph does not need to come from a "nice" equation. In the following example suppose that the vertical value gives the temperature in degrees Celsius and the horizontal value gives the time in hours after midnight on a certain day. It visually shows how the values are related. For example we can see that at 8 a.m. the temperature was 15 degrees Celsius. The temperature was 10 degrees Celsius at two different times: once at 2 a.m. and again at 6 a.m.



DEFINITION OF A FUNCTION

There is a special classification of relationships known as functions. **Functions** are relationships in which each value of the domain corresponds with exactly one value of the range.

Generally x is the variable that we put into an equation to evaluate and find y. For this reason x is considered an input variable and y is considered an output variable. This means the definition of a function, in terms of equations in x and y could be stated as: there is exactly one y value corresponding with any x value.

The box below summarizes this definition and vocabulary.

DEFINITION OF A FUNCTION

A relationship represents a **function** if each input value x is matched with exactly one output value y.

- The **domain** is the set of all input values.
- The **range** is the set of all output values.

Let's revisit some of the examples above to determine whether each relationship represents a function.

- Example 1: Each name is matched with exactly one favorite color so the relationship **is** a function.
- Example 2: This example shows the same relationship as Example 1 but as a set of ordered pairs rather than as a diagram. Notice that no two ordered pairs have the same first component but different second component. So as we already know, the relationship in Example 2 **is** a function.
- Example 3: Alex is wearing two different colors so one input (Alex) is matched with more than one output (red and blue). The relationship **is not** a function.
- Example 5: We saw if $x=0$, then $y=-1$ or $y=1$. Since we know of at least one value of *x* having more than one matching value *y*, the relationship **is not** a function.

Now we will consider some new examples and determine if the relationship represents a function.

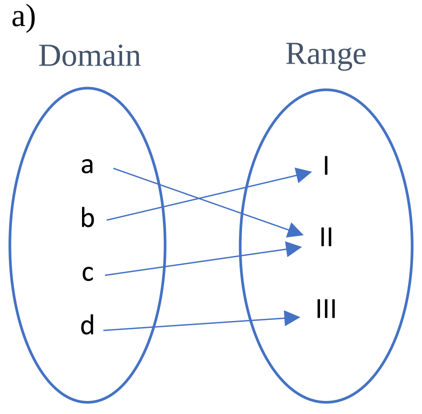
Example 9. Which of the following sets of ordered pairs represents a function?

a) $\{(1,a),(2,b),(3,c),(4,a)\}$ is a function, since there are no two pairs with the same first component. The domain is the set $\{1, 2, 3, 4\}$ and the range is $\{a, b, c\}$.

b) $\{(1,a),(2,b),(1,c),(4,a)\}$ is not a function, since there are two pairs with the same first component but different second components: $(1,a)$ and $(1,c)$.

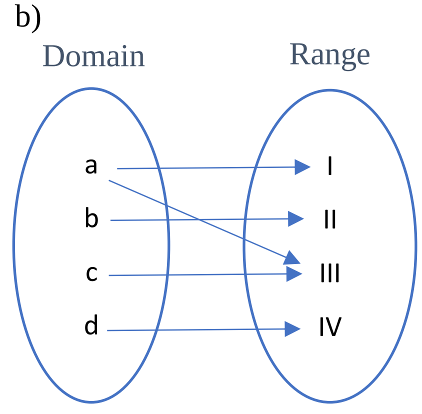
Example 10. Which of the following mapping diagrams represents a function?

a)



This diagram represents a function since each element in the domain corresponds to exactly one element in the range.

b)

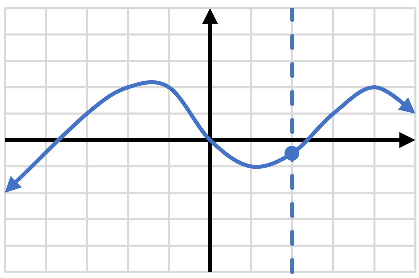


This diagram does not represent a function since one element of the domain corresponds with more than one element of the range; (a,I), (a,III).

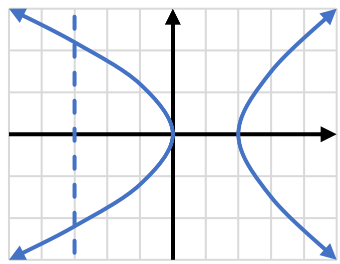
VERTICAL LINE TEST

There is a test called the **Vertical Line Test** to determine if a relationship given graphically is a function. If there is even one vertical line that can be drawn which intersects the graph more than once, then there is an x-value which is matched with more than one y-value, meaning the graph does **not** represent a function.

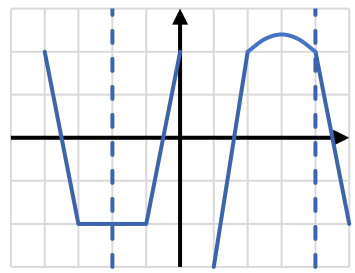
Example 11. Which of the following graphs are graphs of functions?



Drawing a vertical line through this graph will only cross the graph once. It *is* a function.



Drawing a vertical line through this graph will cross the graph twice. This *is not* a function.



Drawing a vertical line through this graph will cross the graph at most once. It *is* a function.

FUNCTION NOTATION

Once we know a relationship represents a function, we often change the notation used to emphasize the fact that it is a function. In accordance with the idea of corresponding input and output values, these equations are often represented with the notation $f(x)$ for the output value. This notation is read as “ f of x ” and signifies the y value that corresponds to a given x value. Writing $y = f(x)$ represents only a change in how the output is referenced. In Example 4 above, instead of writing the function as $y = 3x + 2$, we could have written $f(x) = 3x + 2$.

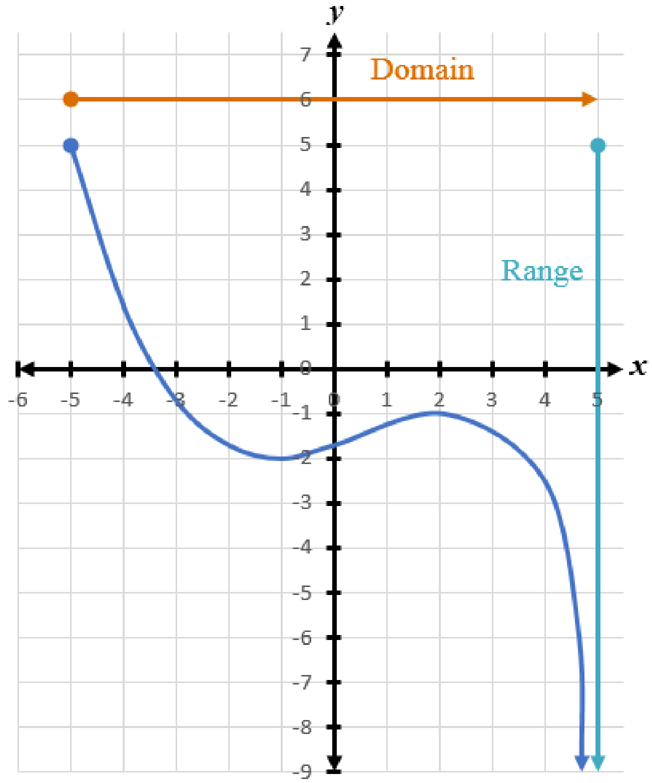
It is important to point out that $f(x)$ does not mean f times x”. It is a notation that names the function with the first letter (function f) and then in parentheses, we are given information about what variable is used as the input in the function (variable x). The first letter naming the function can be anything we want it to be. For example, you will often see $g(x)$, read g of x.

FINDING DOMAIN AND RANGE

We will now identify the domain and range of functions that are represented by graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis.Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values.

Note that the domain and range are always written from smaller to larger values: from left to right for domain, and from the bottom of the graph to the top of the graph for range.

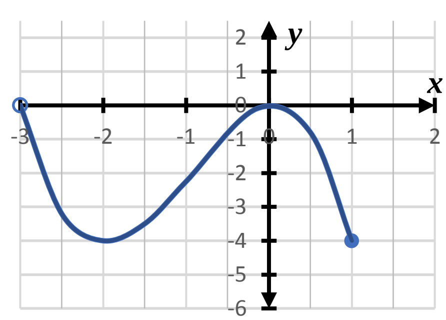
Consider the graph below.



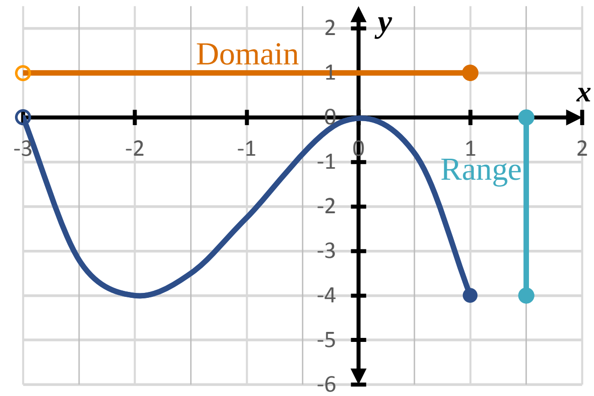
We observe that the graph extends horizontally from -5 to the right without bound, so the domain is "all real numbers greater than or equal to -5". We write that domain in interval notation as $[-5,\infty)$ or in set builder notation as $\{x \mid x \geq -5\}$.

The vertical extent of the graph is all range values 5 and below or "all real numbers less than or equal to 5". So the range is written in interval notation as $(-\infty,5]$ or in set builder notation as $\{y \mid y \leq 5\}$.

Example 12. Find the domain and range given the following graph.



Answer:



Domain: $(-3,1]$ or $-3 < x \le 1$

Range: $[-4,0]$ or $-4 \le y \le 0$

Notice that the open circle at $(-3,0)$ indicates that the point $(-3,0)$ does not belong to the graph. The closed circle at $(1,-4)$ indicates that the point $(1,-4)$ does belong to the graph.

Example 13. Find the domain of the function.

$f(x) = \dfrac{3x-1}{x^2+x-6}$	Find values of x for which the denominator is equal to 0.
$x^2 + x - 6 = 0$	Solve by factoring
$(x+3)(x-2) = 0$	Set each factor equal to zero
$x+3=0$ or $x-2=0$	Solve each equation
$\underline{-3 \quad -3}$ $\qquad$ $\underline{+2 \quad +2}$	
$x=-3$ or $x=2$	Exclude these values from the domain of f
All real numbers except $x=-3$ and 2	Our Answer

The notation in the previous example tells us that x can be any value except for -3 and -2. If x were one of those two values, the function would be undefined.

Example 14. Find the domain of the function.

$f(x) = 3x^2 - x$	With this function, there are no excluded values
All real numbers or $\mathbb{R}$ or $(-\infty,\infty)$	Our Solution

In the above example there are no real numbers that make the function undefined. This means any number can be used for x.

Example 15. Find the domain of the function.

$f(x) = \sqrt{2x-3}$	Square roots of negative numbers are not real numbers, so we restrict the domain to all real numbers for which the radicand is nonnegative

$2x - 3 \geq 0$	Set up an inequality
$\underline{\quad +3 \quad +3}$	Solve
$\frac{2x}{2} \geq \frac{3}{2}$	
$x \geq \frac{3}{2}$	
$x \geq \frac{3}{2}$	Our Answer

The notation in the above example states that our variable can be any number greater than or equal to $\frac{3}{2}$. Any number smaller than $\frac{3}{2}$ would make the function undefined because the radicand would have a value less than zero.

EVALUATING A FUNCTION USING FUNCTION NOTATION

Function notation can be used when we want to evaluate a function. A numerical value or an expression is substituted in the place of the input variable in the equation and the output value or expression is determined. This process is shown in the following examples.

Example 16. Evaluate the function.

Let $f(x) = 3x^2 - 4x$; find $f(-2)$.	Substitute -2 for x in the function
$f(-2) = 3(-2)^2 - 4(-2)$	Evaluate, using order of operations
$= 3(4) - 4(-2)$	Multiply
$= 12 + 8$	Add
$f(-2) = 20$	Our Answer

One advantage of using function notation is that instead of asking "what is the value of y if $x = -2$?", we can now just write "Find $f(-2)$". The answer statement $f(-2) = 20$ above tells us the value of the function is 20 when $x = -2$.

Example 17. Evaluate the function.

Let $h(x) = 3^{2x-6}$; find $h(4)$.	Substitute 4 for x in the function
$h(4) = 3^{2(4)-6}$	Simplify exponent, multiplying first
$= 3^{8-6}$	Subtract in exponent
$= 3^2$	Evaluate exponent
$h(4) = 9$	Our Answer

Example 18. Evaluate the function.

Let $k(a)=2\|a+4\|$; find $k(-7)$.	Substitute -7 for a in the function
$k(-7)=2\|-7+4\|$	Add inside absolute value
$=2\|-3\|$	Evaluate absolute value
$=2(3)$	Multiply
$k(-7)=6$	Our Answer

As the above examples show, the function can take many different forms, but the way to evaluate the function is always the same: replace the variable with what is in parentheses and simplify.

We can also substitute expressions into functions using the same process. Often the expressions use the same variable, so it is important to remember each occurrence of the variable is replaced by whatever is in parentheses.

Example 19. Evaluate the function.

Let $g(x)=x^4+1$; find $g(3x)$.	Replace x in the function with $(3x)$
$g(3x)=(3x)^4+1$	Simplify exponent
$g(3x)=81x^4+1$	Our Answer

Example 20. Evaluate the function.

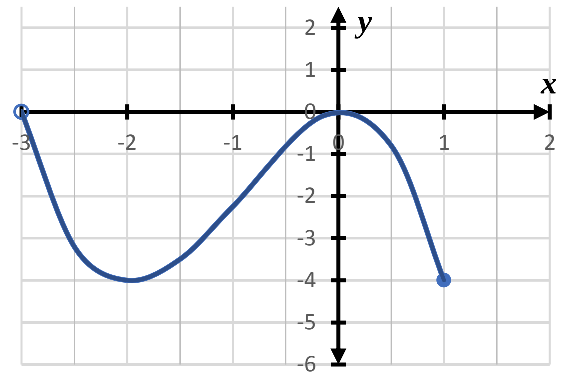
Let $p(t)=t^2-t$; find $p(t+1)$.	Replace each t in the function with $(t+1)$
$p(t+1)=(t+1)^2-(t+1)$	Square binomial
$=t^2+2t+1-(t+1)$	Distribute negative sign
$=t^2+2t+1-t-1$	Combine like terms
$p(t+1)=t^2+t$	Our Answer

It is important to become comfortable with function notation and learn how to use it as we transition into more advanced algebra topics.

EVALUATING A FUNCTION FROM ITS GRAPH

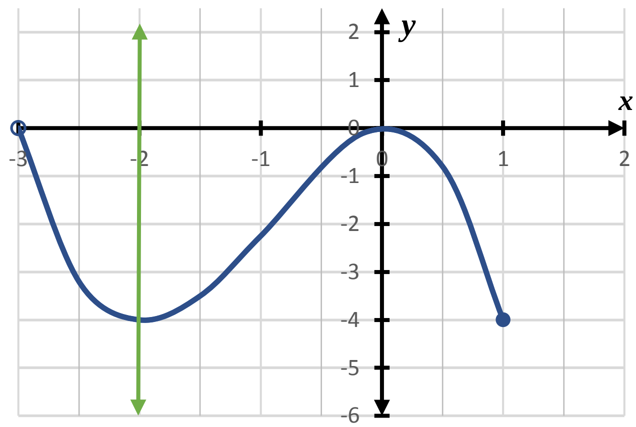
Finally, in addition to being evaluated algebraically, functions can be evaluated by identifying input and output values on its graph. To do this, we only need to find the desired value for the input variable on the *x*-axis, and then moving along a vertical path through that value, note where this path intersects the graph of the function. Upon finding this intersection we now should move along a horizontal path toward the y-axis. The value at which this horizontal path intersects the *y*-axis represents the output value for the desired input value.

Example 21. Using the graph of the function *f* below, find $f(-2)$.



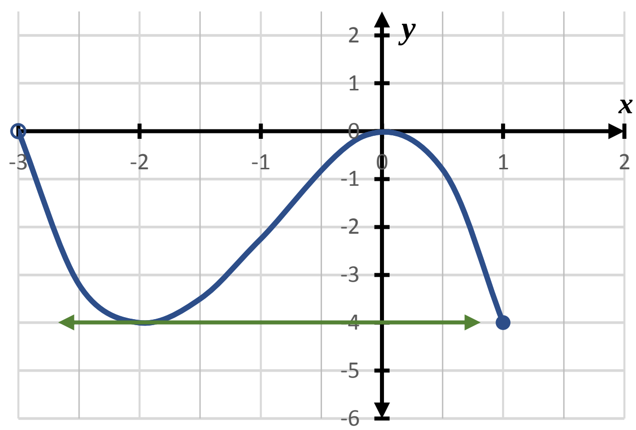
Locate -2 on the *x*-axis and visualize a vertical path passing through that value.

Remember that this is where the input value is equal to -2.

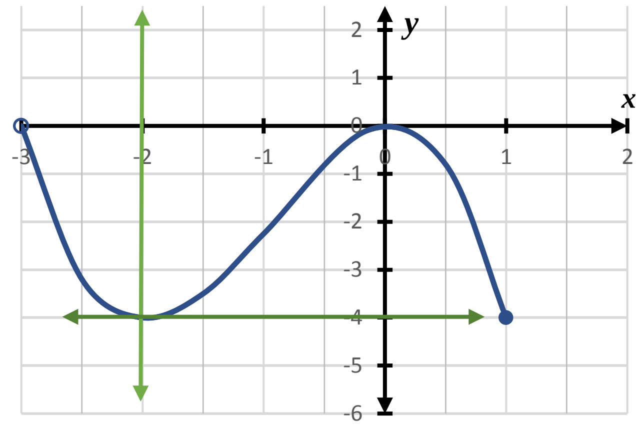


Now, notice where this line intersects the graph of the given function.

At that point of intersection, visualize a horizontal line passing through and in the direction of the y-axis.



At this point it should be noted that the horizontal line intersects the y-axis at -4. This is the value of the output variable. Therefore, visualizing both the vertical and horizontal lines at the same time identifies the values of both the input and output variables, and in effect finds what we are looking for, $f(-2)$. This is expressly seen in the graph on the next page.

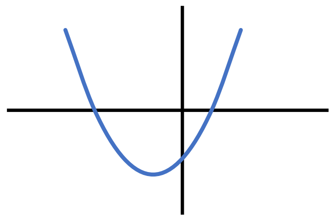


Therefore, according to the procedure followed, and the vertical and horizontal lines, we see that $f(-2) = -4$. In other words, the function evaluated at -2 is equal to -4.

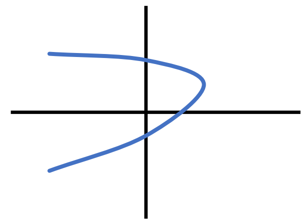
Practice Exercises
Section 5.1: Functions

Determine which of the following represent functions.

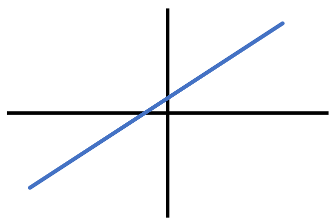
1)



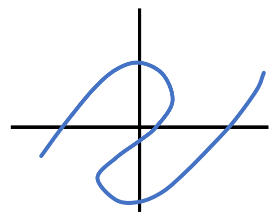
3)



2)



4)



Specify the domain of each of the following functions.

5) $f(x) = -5x + 1$

6) $f(x) = \sqrt{8 - 4x}$

7) $s(t) = \dfrac{1}{t^2}$

8) $s(t) = \dfrac{1}{t^2 + 1}$

9) $f(x) = x^2 - 3x - 4$

10) $f(x) = \sqrt{x - 16}$

11) $f(x) = \dfrac{-2}{x^2 - 3x - 4}$

12) $y(x) = \dfrac{x}{x^2 - 25}$

Evaluate each function.

13) $g(x) = 3x^2 + 4x - 4$; Find $g(0)$

14) $g(x) = 5x - 3$; Find $g(2)$

15) $f(x) = 3x + 1$; Find $f(0)$

16) $f(x) = 2x^2 + 9x + 4$; Find $f(-9)$

17) $f(n) = n^2 - 9n - 3$; Find $f(10)$

18) $f(t) = 3^t - 2$; Find $f(4)$

19) $w(x) = x^2 + 4x$; Find $w(-5)$

20) $w(n) = 4n + 3$; Find $w(2)$

21) $p(n) = -3|n|$; Find $p(-7)$

22) $h(n) = 4n + 2$; Find $h(n + 3)$

23) $g(x) = x + 1$; Find $g(3x)$

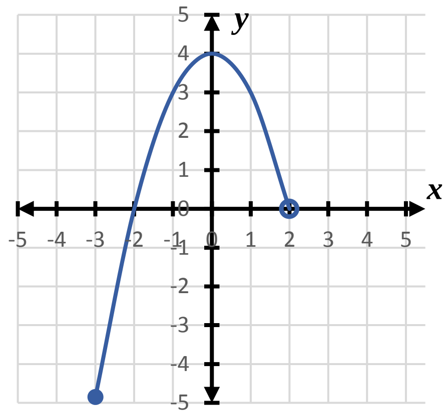
24) $h(t) = t^2 + t$; Find $h(t + 1)$

Practice Exercises: Section 5.1 (continued)

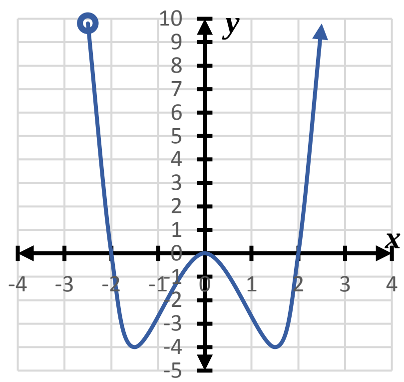
For each of the functions in problems 25 - 27, find the following:

a) Domain
b) Range
c) $f(-1)$
d) $f(0)$
e) the x value(s) such that $f(x)=0$

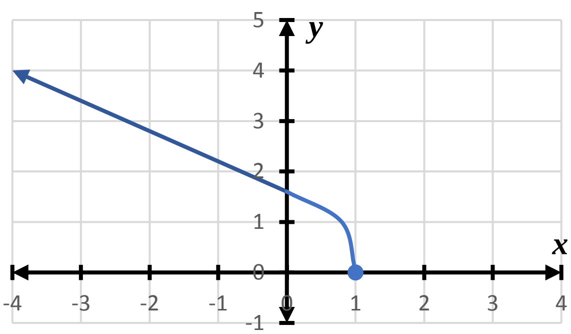
25)



26)



27)



Determine whether each relation is a function. Identify the domain and range for each relation.

28) $\{(-2,4),(3,6),(4,6),(9,0)\}$

29) $\{(0,-5),(2,-3),(4,-1),(6,1),(8,3)\}$

30) $\{(5,6),(7,0),(9,-1),(7,1)\}$

ANSWERS to Practice Exercises
Section 5.1: Functions

1) Yes
2) Yes
3) No
4) No

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5) $(-\infty, \infty)$
6) $(-\infty, 2]$
7) all real numbers except $t = 0$
8) $(-\infty, \infty)$
9) $(-\infty, \infty)$
10) $[16, \infty)$
11) all real numbers except $x = -1$ and $x = 4$
12) all real numbers except $x = -5$ and $x = 5$

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13) -4
14) 7
15) 1
16) 85
17) 7
18) 79
19) 5
20) 11
21) -21
22) $4n + 14$
23) $3x + 1$
24) $t^2 + 3t + 1$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 5.1 (continued)

25)
a) $-3 \le x < 2$
b) $-5 \le y \le 4$
c) $f(-1) = 3$
d) $f(0) = 4$
e) $x = -2$

26)
a) $x > -2.5$
b) $y \ge -4$
c) $f(-1) = -3$
d) $f(0) = 0$
e) $x = -2, 0, 2$

27)
a) $x > -2.5$
b) $y \ge -4$
c) $f(-1) = -3$
d) $f(0) = 1.5$
e) $x = 1$

28)
Function;
Domain $\{-2, 3, 4, 9\}$
Range $\{0, 4, 6\}$

29)
Function;
Domain $\{0, 2, 4, 6, 8\}$
Range $\{0, 4, 6\}$

30)
Not a Function;
Domain $\{5, 7, 9\}$
Range $\{-1, 0, 1, 6\}$