Section 5.3: Exponential Functions and Equations

Objectives: Graph exponential functions.
Solve exponential equations by finding a common base.

As our study of algebra gets more advanced, we begin to study more involved functions. One pair of inverse functions we will look at are exponential functions and logarithmic functions. Here we will look at exponential functions and then we will consider logarithmic functions in another section.

GRAPHING EXPONENTIAL FUNCTIONS

Exponential functions have the form $f(x)=b^x$ where $b>0$ and $b\neq 1$. Notice that exponential functions have the variable in the exponent. It is important not to confuse exponential functions with polynomial functions where the variable is in the base such as $f(x)=x^2$.

Example 1.

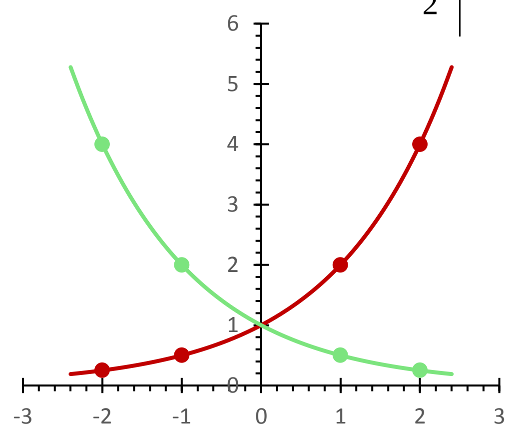
Evaluate $f(x)=2^x$ and $g(x)=\left(\frac{1}{2}\right)^x$ at $x=-2,-1,0,1,2$ and graph.

$f(x)=2^x$

x	$y=f(x)$
-2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}$
-1	$2^{-1}=\frac{1}{2^1}=\frac{1}{2}$
0	$2^0=1$
1	$2^1=2$
2	$2^2=4$

$g(x)=\left(\frac{1}{2}\right)^x$

x	$y=g(x)$
-2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$
-1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$
0	$\left(\frac{1}{2}\right)^0=1$
1	$\left(\frac{1}{2}\right)^1=\frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$



The domain of $f(x) = b^x$ consists of all real numbers, written in interval notation as $(-\infty, \infty)$. The range of $f(x) = b^x$ consists of all positive real numbers, written in interval notation as $(0, \infty)$.

If $b > 1$, then the graph goes up to the right and is an increasing function, called an exponential **growth** function. If $0 < b < 1$, then the graph goes down to the right and is a decreasing function, called an exponential **decay** function.

The graphs of all exponential functions of the form $f(x) = b^x$ pass through the point $(0,1)$.

The graph of an exponential function $f(x) = b^x$ approaches, but does not touch, the x-axis. We say the x-axis, or the line $y = 0$, is a *horizontal asymptote* of the graph of the function.

SOLVING EXPONENTIAL EQUATIONS

Solving exponential equations cannot be done using the skill set we have seen in the past. For example, if $3^x = 9$, we cannot take the x^{th}-root of 9 because we do not know what the index is and this doesn't get us any closer to finding x. However, we may notice that 9 is 3^2. We can then conclude that if $3^x = 3^2$ then $x = 2$.

We will use this process to solve exponential equations. If we can rewrite an equation so that the bases match, then the exponents must also match.

Example 2. Solve the equation.

$4^{2x} = 4^{x+3}$	Same bases, set exponents equal
$2x = x + 3$	Solve
$2x = x + 3$	Subtract x from both sides
$\underline{-x \quad -x}$	
$x = 3$	Our Solution

Example 3. Solve the equation.

$5^{2x+1} = 125$	Rewrite 125 as 5^3
$5^{2x+1} = 5^3$	Same bases, set exponents equal
$2x + 1 = 3$	Solve
$\underline{-1 \quad -1}$	Subtract 1 from both sides
$\dfrac{2x}{2} = \dfrac{2}{2}$	Divide both sides by 2
$x = 1$	Our Solution

Sometimes we may have to do work on both sides of the equation to get a common base. As we do so, we will use various exponent properties to help. First we will use the exponent property that states $(a^x)^y = a^{xy}$.

Example 4. Solve the equation.

$8^{3x} = 32$	Rewrite 8 as 2^3 and 32 as 2^5
$(2^3)^{3x} = 2^5$	Multiply exponents 3 and $3x$
$2^{9x} = 2^5$	Same bases, set exponents equal
$\frac{9x}{9} = \frac{5}{9}$	Solve Divide both sides by 9
$x = \frac{5}{9}$	Our Solution

As we multiply exponents, we may need to distribute if there are several terms involved.

Example 5. Solve the equation.

$3^{3x+5} = 81^{4x+1}$	Rewrite 81 as 3^4
$3^{3x+5} = (3^4)^{4x+1}$	Multiply exponents $4(4x+1)$ by distributing
$3^{3x+5} = 3^{16x+4}$	Same bases, set exponents equal
$3x + 5 = 16x + 4$	Solve
$-16x \quad -16x$	Subtract $16x$ from both sides
$-13x + 5 = 4$	
$-5 \quad -5$	Subtract 5 from both sides
$\frac{-13x}{-13} = \frac{-1}{-13}$	Divide both sides by -13
$x = \frac{1}{13}$	Our Solution

Another useful property of exponents is that negative exponents will give us a reciprocal: $a^{-n} = \frac{1}{a^n}$.

Example 6. Solve the equation.

$\left(\frac{1}{9}\right)^{2x} = 3^{7x-1}$	Rewrite $\frac{1}{9}$ as 3^{-2} (Use the negative exponent property)
$(3^{-2})^{2x} = 3^{7x-1}$	Multiply exponents -2 and $2x$
$3^{-4x} = 3^{7x-1}$	Same bases, set exponents equal

$-4x = 7x - 1$	Subtract $7x$ from both sides
$\underline{-7x \quad -7x}$	
$\frac{-11x}{-11} = \frac{-1}{-11}$	Divide by both sides by -11
$x = \frac{1}{11}$	Our Solution

If we have several factors with the same base on one side of the equation, we can add the exponents using the property that states $a^x a^y = a^{x+y}$.

Example 7. Solve the equation.

$5^{4x} \cdot 5^{2x-1} = 5^{3x+11}$	Add exponents on left, combing like terms
$5^{6x-1} = 5^{3x+11}$	Same bases, set exponents equal
$6x - 1 = 3x + 11$	Solve
$\underline{-3x \qquad -3x}$	Subtract $3x$ from both sides
$3x - 1 = 11$	Add 1 to both sides
$\underline{+1 \quad +1}$	
$\frac{3x}{3} = \frac{12}{3}$	Divide both sides by 3
$x = 4$	Our Solution

It may take a bit of practice to get used to knowing which base to use. We will use our properties of exponents to help us simplify. Again, below are the properties we used:

$$(a^x)^y = a^{xy} \quad \text{and} \quad \frac{1}{a^n} = a^{-n} \quad \text{and} \quad a^x a^y = a^{x+y}$$

In all of the equation we solved here, we were able to find a common base. However, this is not always possible. For example, $2 = 10^x$ cannot be written using a common base. To solve equations like this, we will need to use the inverse of an exponential function. The inverse is called a logarithmic function, which we will discuss in another section.

Practice Exercises
Section 5.3: Exponential Functions and Equations

Graph the following exponential functions and write the domain and range in interval notation.

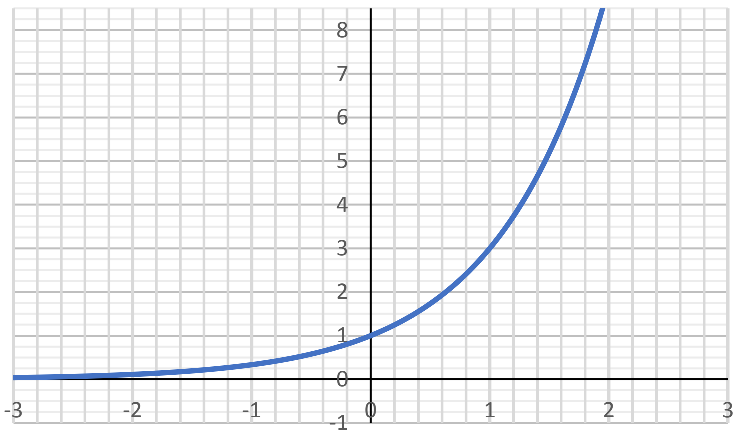
1. $f(x) = 3^x$
2. $f(x) = \left(\frac{1}{3}\right)^x$
3. $f(x) = 2^x + 3$

Solve.

4. $3^{1-2n} = 3^{1-3n}$
5. $4^{2x} = \frac{1}{16}$
6. $4^{2a} = 1$
7. $16^{-3p} = 64^{-3p}$
8. $\left(\frac{1}{25}\right)^{-k} = 125^{-2k-2}$
9. $625^{-n-2} = \frac{1}{125}$
10. $6^{2m+1} = \frac{1}{36}$
11. $6^{2r-3} = 6^{r-3}$
12. $6^{-3x} = 36$
13. $5^{2n} = 5^{-n}$
14. $64^b = 2^5$
15. $216^{-3v} = 36^{3v}$
16. $\left(\frac{1}{4}\right)^x = 16$
17. $27^{-2n-1} = 9$
18. $4^{3a} = 4^3$
19. $4^{-3v} = 64$
20. $64^{x+2} = 16$
21. $9^{2n+3} = 243$
22. $16^{2k} = \frac{1}{64}$
23. $243^p = 27^{-3p}$
24. $3^{-2x} = 3^3$
25. $4^{2n} = 4^{2-3n}$
26. $5^{m+2} = 5^{-m}$
27. $625^{2x} = 25$
28. $\left(\frac{1}{36}\right)^{b-1} = 216$
29. $216^{2n} = 36$
30. $6^2 \cdot 6^{2x} = 6^2$
31. $3^{2m} \cdot 3^4 = 1$
32. $2^{3x} \cdot 2^{5x} = 2^{4x+12}$

ANSWERS to Practice Exercises
Section 5.3: Exponential Functions and Equations

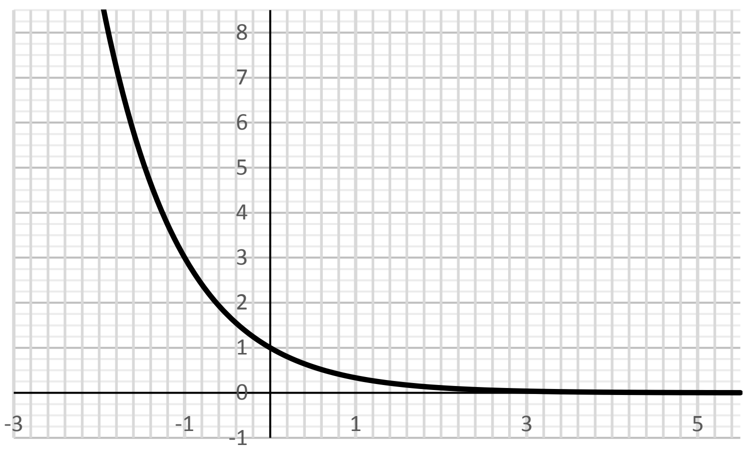
1)



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

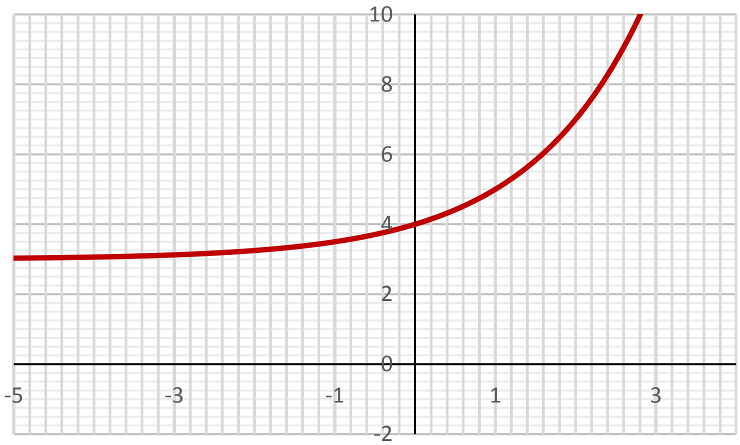
2)



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

3)



Domain: $(-\infty, \infty)$

Range: $(3, \infty)$

ANSWERS to Practice Exercises: Section 5.3 (continued)

4) 0

5) -1

6) 0

7) 0

8) $-\frac{3}{4}$

9) $-\frac{5}{4}$

10) $-\frac{3}{2}$

11) 0

12) $-\frac{2}{3}$

13) 0

14) $\frac{5}{6}$

15) 0

16) -2

17) $-\frac{5}{6}$

18) 1

19) -1

20) $-\frac{4}{3}$

21) $-\frac{1}{4}$

22) $-\frac{3}{4}$

23) 0

24) $-\frac{3}{2}$

25) $\frac{2}{5}$

26) -1

27) $\frac{1}{4}$

28) $-\frac{1}{2}$

29) $\frac{1}{3}$

30) 0

31) -2

32) 3