# Section 5.4: Logarithmic Functions and Equations

**Objectives: Convert between logarithms and exponents and use that relationship to solve basic logarithmic equations.**
**Graph logarithmic functions.**

The inverse of an exponential function is a function known as a logarithm. Logarithms are studied in detail in advanced algebra and science courses. Some places where logarithms arise in science are in measuring the *ph*-level of a chemical, measuring the intensity of an earthquake using the Richter scale, and measuring the intensity of a sound using decibels

## DEFINITION OF THE LOGARITHMIC FUNCTION

Here, we will take an introductory look at how logarithms work. When working with radicals, we found that there were two ways to write radicals. The expression $\sqrt[n]{a^m}$ could be written as $a^{\frac{m}{n}}$. Each form has its advantages, thus we need to be comfortable using both the radical form and the rational exponent form.

Similarly, an exponent can be written in two forms, each with its own advantages. We are familiar with the first form, $b^y = x$, where $b$ is the base, $x$ can be thought of as our answer, and $y$ is the exponent. The second way to write this is in logarithm form as $\log_b x = y$. The word "log" tells us that we are in this form. The parts of the equation all still mean the same thing: $b$ is the base, $x$ can be thought of as our answer, and $y$ is the exponent.

**LOGARITHMIC FUNCTION WITH BASE** $b$

For $x > 0$, $b > 0$, and $b \neq 1$,

$$y = \log_b x \qquad \text{if and only if} \qquad x = b^y$$

The function given by $f(x) = \log_b x$ read as "log base $b$ of $x$" is called the *logarithmic function with base* $b$.

Notice a logarithm is an exponent. Thus, logarithmic form will let us isolate an exponent. Using this idea, the equation $5^2 = 25$ could also be written in equivalent logarithmic form as $\log_5 25 = 2$. Both mean the same thing, both are still the same exponent problem, but each form has its own advantages. The most important thing to be comfortable doing with logarithms and exponents is to be able to switch back and forth between the two forms. This process is shown in the next few examples.

## CONVERTING BETWEEN EXPONENTIAL AND LOGARITHMIC FORM

**Example 1**. In each part, write the exponential equation in its equivalent logarithmic form.

A. $2 = 10^x$ Identify base $10$, answer $2$, and exponent $x$

$\log_{10} 2 = x$ Our Answer

B. $m^3 = 5$ Identify base $m$, answer $5$, and exponent $3$

$\log_m 5 = 3$ Our Answer

C. $7^2 = b$ Identify base $7$, answer $b$, and exponent $2$

$\log_7 b = 2$ Our Answer

D. $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$ Identify base $\frac{2}{3}$, answer $\frac{16}{81}$, and exponent $4$

$\log_{\frac{2}{3}} \frac{16}{81} = 4$ Our Answer

**Example 2**. In each part, write the logarithmic equation in its equivalent exponential form.

A. $\log_4 16 = 2$ Identify base $4$, answer $16$, and exponent $2$

$4^2 = 16$ Our Answer

B. $\log_3 x = 7$ Identify base $3$, answer $x$, and exponent $7$

$3^7 = x$ Our Answer

C. $\log_9 3 = \frac{1}{2}$ Identify base $9$, answer $3$, and exponent $\frac{1}{2}$

$9^{\frac{1}{2}} = 3$ Our Answer

## EVALUATING LOGARITHMS

Once we are comfortable switching between logarithmic and exponential form, we are able to evaluate logarithmic expressions and solve exponential equations. We will first evaluate logarithmic expressions. An easy way to evaluate a logarithm is to set the logarithm equal to $x$ and change it into an exponential equation.

**Example 3**. Evaluate.

| | |
|---|---|
| Evaluate $\log_2 64$ | Set logarithm equal to $x$ |
| $\log_2 64 = x$ | Change to exponential form |
| $2^x = 64$ | Write using same bases, $64 = 2^6$ |
| $2^x = 2^6$ | Same bases, set exponents equal |
| $x = 6$ | Our Answer |

**Example 4**. Evaluate.

| | |
|---|---|
| Evaluate $\log_{125} 5$ | Set logarithm equal to $x$ |
| $\log_{125} 5 = x$ | Change to exponential form |
| $125^x = 5$ | Write using same bases, $125 = 5^3$ |
| $(5^3)^x = 5$ | Multiply exponents |
| $5^{3x} = 5$ | Same bases, set exponents equal ($5 = 5^1$) |
| $\frac{3x}{3} = \frac{1}{3}$ | Solve<br>Divide both sides by $3$ |
| $x = \frac{1}{3}$ | Our Answer |

**Example 5**. Evaluate.

| | |
|---|---|
| Evaluate $\log_3 \frac{1}{27}$ | Set logarithm equal to $x$ |
| $\log_3 \frac{1}{27} = x$ | Change to exponential form |
| $3^x = \frac{1}{27}$ | Write using same bases, $\frac{1}{27} = 3^{-3}$ |
| $3^x = 3^{-3}$ | Same bases, set exponents equal |
| $x = -3$ | Our Answer |

## SOLVING LOGARITHMIC EQUATIONS

Solving logarithmic equations is done in a very similar way, by changing the equation into exponential form and solving the resulting equation.

**Example 6**. Solve the equation.

| | |
|---|---|
| $\log_5 x = 2$ | Change to exponential form |
| $5^2 = x$ | Evaluate exponent |
| $25 = x$ | Our Solution |

**Example 7**. Solve the equation.

| | |
|---|---|
| $\log_2(3x+5)=4$ | Change to exponential form |
| $2^4=3x+5$ | Evaluate exponent |
| $16=3x+5$ | Solve |
| $\underline{-5 \quad -5}$ | Subtract $5$ from both sides |
| $\frac{11}{3}=\frac{3x}{3}$ | Divide both sides by $3$ |
| $\frac{11}{3}=x$ | Our Solution |

**Example 8**. Solve the equation.

| | |
|---|---|
| $\log_x 8=3$ | Change to exponential form |
| $x^3=8$ | Cube root of both sides |
| $x=2$ | Our Solution |

## COMMON LOGARITHM AND NATURAL LOGARITHM

There is one base of the logarithm that is used more often than any other base, mainly base 10. Similar to square roots and not writing the common index of 2 in the radical, we don't write the common base of 10 in the logarithm. So if we are working on a problem with no base written, we will always assume that the base is 10.

The other important log is the "natural", or "log base $e$", denoted as "$\ln(x)$" and usually pronounced as "ell-enn-of-$x$". (Note: That's "ell-enn", not "one-enn" or "eye-enn"!) Just as the number $e$ arises naturally in math and the sciences, so also does the natural log, which is why you need to be familiar with it.

The number $e$ is a constant approximately equal to 2.72. The number $e$ is a constant similar in idea to $\pi$ in that it goes on forever without repeat or pattern. Just as $\pi$ naturally occurs in several geometry applications, $e$ appears in many exponential applications which we will see in the next section.

**Example 9**. Solve the equation.

| | |
|---|---|
| $\log x=-2$ | Rewrite as exponent, $10$ is base |
| $10^{-2}=x$ | Evaluate. Remember a negative exponent gives a fraction |
| $\frac{1}{100}=x$ | Our Solution |

**Example 10**. Solve the equation.

| | |
|---|---|
| $\ln x = 3$ | Rewrite as exponent, $e$ is base |
| $e^3 = x$ | Our Solution |

So far, this lesson has introduced the idea of logarithms, changing between logarithms and exponents, evaluating logarithms, and solving basic logarithmic equations. In an advanced algebra course, logarithms will be studied in much greater detail.

## GRAPHING LOGARITHMIC FUNCTIONS

It was mentioned in the previous section that logarithmic functions are inverses of exponential functions. To get an understanding of the graph of $f(x) = \log_b x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

The domain of $f(x) = \log_b x$ consists of all positive real numbers, $(0, \infty)$.

The range of $f(x) = \log_b x$ consists of all real numbers, $(-\infty, \infty)$.

The graph below shows the exponential function $f(x) = 2^x$ and its inverse function $g(x) = \log_2 x$.

**Example 11**. Graph $f(x) = \log_3 x$.

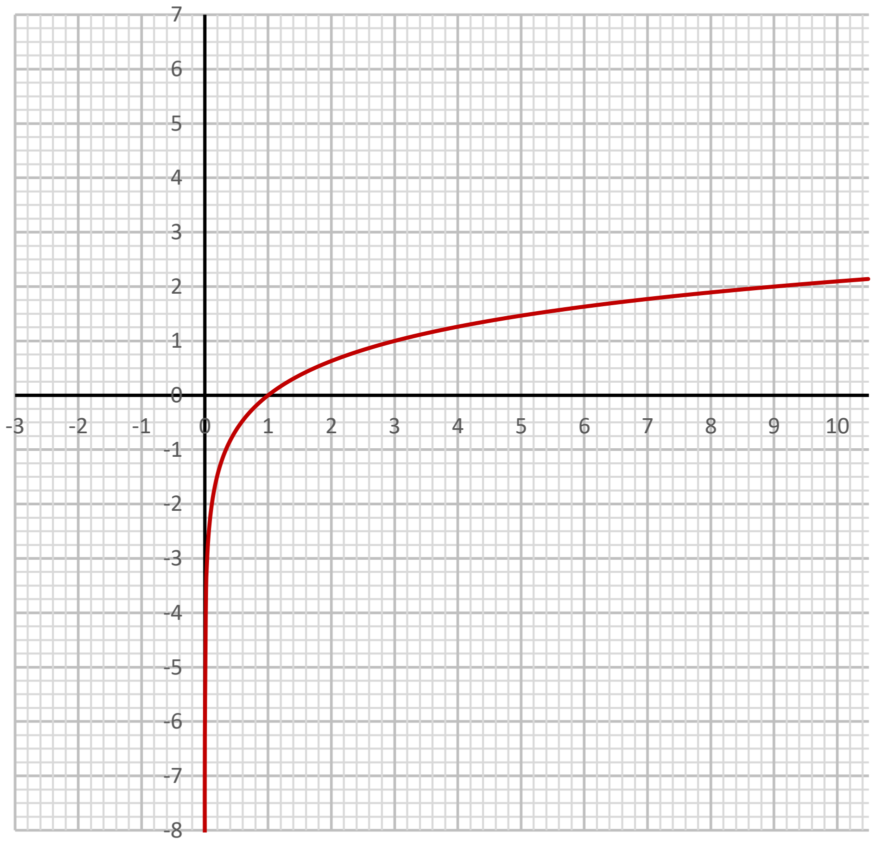
Use the definition from earlier in the section:

For $x > 0$, $b > 0$, and $b \neq 1$, $y = \log_b x$ if and only if $x = b^y$.

Rewrite $f(x) = \log_3 x$ in exponential form as $3^{f(x)} = x$ or $3^y = x$.

Evaluate $3^y = x$ for selected *y*-values $y = -2, -1, 0, 1, 2$ and graph.

| $x$ | $y = f(x)$ |
|---|---|
| $3^{-2} = \frac{1}{9}$ | -2 |
| $3^{-1} = \frac{1}{3}$ | -1 |
| $3^0 = 1$ | 0 |
| $3^1 = 3$ | 1 |
| $3^2 = 9$ | 2 |




Written in interval notation, he domain of $f(x) = \log_3 x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

# Practice Exercises
## Section 5.4: Logarithmic Functions and Equations

**Rewrite each equation in exponential form.**

1. $\log_9 81 = 2$
2. $\ln a = -16$
3. $\log_7 \frac{1}{49} = -2$
4. $\log 10,000 = 4$
5. $\log_{13} 169 = 2$
6. $\ln 1 = 0$

**Rewrite each equation in logarithmic form.**

7. $8^0 = 1$
8. $17^{-2} = \frac{1}{289}$
9. $15^2 = 225$
10. $144^{\frac{1}{2}} = 12$
11. $64^{\frac{1}{6}} = 2$
12. $19^2 = 361$

**Evaluate each expression.**

13. $\log_{125} 5$
14. $\log_5 125$
15. $\log_{343} \frac{1}{7}$
16. $\log_7 1$
17. $\log_4 16$
18. $\log_4 \frac{1}{64}$
19. $\log_6 36$
20. $\log_{36} 6$
21. $\log_2 64$
22. $\log_3 243$

**Solve each equation.**

23. $\log_5 x = 1$
24. $\log_8 k = 3$
25. $\log_2 x = -2$
26. $\log n = 3$

*The Practice Exercises are continued on the next page.*

*Practice Exercises: Section 5.4 (continued)*

**Solve each equation.**

27. $\log_{11} k = 2$
28. $\log_4 p = 4$
29. $\log_9 (n+9) = 4$
30. $\log_{11} (x-4) = -1$
31. $\log_5 (-3m) = 3$
32. $\log_2 (-8r) = 1$
33. $\log_{11} (x+5) = -1$
34. $\log_7 (-3n) = 4$
35. $\log_4 (6b+4) = 0$
36. $\log_{11} (10v+1) = -1$
37. $\log_5 (-10x+4) = 4$
38. $\log_9 (7-6x) = -2$
39. $\log_2 (10-5a) = 3$
40. $\log_8 (3k-1) = 1$

**Graph each logarithmic function.**

41. $f(x) = \log_2 x$
42. $f(x) = \log_3 (x-1)$

# ANSWERS to Practice Exercises
## Section 5.4: Logarithmic Functions and Equations

1) $9^2 = 81$
2) $e^{-16} = a$
3) $7^{-2} = \frac{1}{49}$
4) $10^4 = 10,000$
5) $13^2 = 169$
6) $e^0 = 1$

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7) $\log_8 1 = 0$
8) $\log_{17} \frac{1}{289} = -2$
9) $\log_{15} 225 = 2$
10) $\log_{144} 12 = \frac{1}{2}$
11) $\log_{64} 2 = \frac{1}{6}$
12) $\log_{19} 361 = 2$

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13) $\frac{1}{3}$
14) 3
15) $-\frac{1}{3}$
16) 0
17) 2
18) $-3$
19) 2
20) $\frac{1}{2}$
21) 6
22) 5

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23) 5
24) 512
25) $\frac{1}{4}$
26) 1000

*The Answers to Practice Exercises are continued on the next page.*

*ANSWERS to Practice Exercises: Section 5.4 (continued)*

27) 121
28) 256
29) 6552
30) $\frac{45}{11}$
31) $-\frac{125}{3}$
32) $-\frac{1}{4}$
33) $-\frac{54}{11}$
34) $-\frac{2401}{3}$
35) $-\frac{1}{2}$
36) $-\frac{1}{11}$
37) $-\frac{621}{10}$
38) $\frac{283}{243}$
39) $\frac{2}{5}$
40) 3

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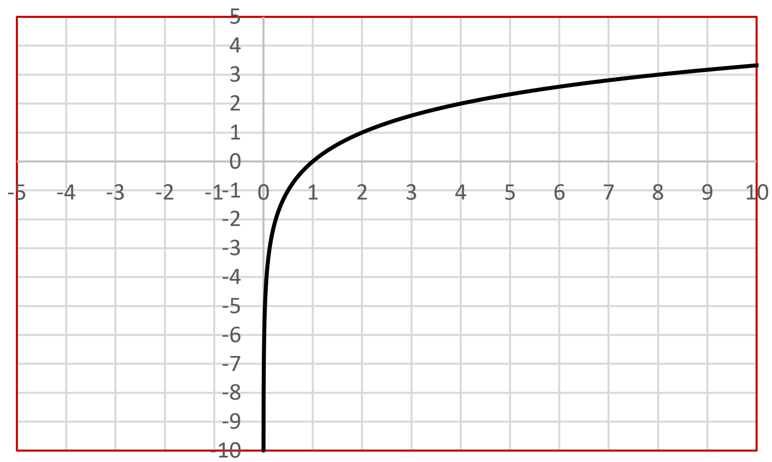
41)

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