Section 5.5: Compound Interest

Objective: Calculate final account balances using the formulas for compound and continuous interest.

COMPOUND INTEREST

One application of exponential functions involves compound interest. When money is invested in an account (or borrowed as a loan), a certain amount is added to the balance. This money added to the balance is called the *interest*. Once that interest is added to the balance, it will earn more interest during the next compounding period. This idea of earning interest on interest is called compound interest.

As an example, suppose you invest $100 at 10% interest compounded annually:

> After one year you will earn $10 in interest, giving you a new balance of $110.
>
> The next year you will earn 10% of $110 or $11, giving you a new balance of $121.
>
> The third year you will earn 10% of $121 or $12.10, giving you a new balance of $133.10.

This pattern will continue each year until you close the account.

There are several ways interest can be paid. The first way, as described above, is compounded annually. In this model, the interest is paid once per year. But interest can be compounded more often. Some common compounding periods include semi-annually (twice per year), quarterly (four times per year, such as quarterly taxes), monthly (12 times per year, such as a savings account), weekly (52 times per year), or even daily (365 times per year, such as some student loans). When interest is compounded in any of these ways, we can calculate the balance after any amount of time using the following formula:

COMPOUND INTEREST FORMULA

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = final amount
P = principal (starting balance)
r = annual interest rate (as a decimal)
n = number of compounding periods per year
t = time (in years)

Example 1.

If you take a car loan for $25,000 with an annual interest rate of 6.5% compounded quarterly with no payments required for the first five years, what will your balance be at the end of those five years?

Identify information given:

$P = 25,000$, $r = 0.065$, $n = 4$, $t = 5$

$A = 25000\left(1 + \frac{0.065}{4}\right)^{4\cdot 5}$ Plug each value into formula, evaluate parentheses

$A = 25000(1.01625)^{4\cdot 5}$ Multiply exponents

$A = 25000(1.01625)^{20}$ Evaluate exponent

$A = 25000(1.38041977\ldots)$ Multiply

$A = 34510.49$

$34, 510.49 Our Answer

Notice that we did not round any of the numbers until the final answer (which we rounded to the nearest cent).

We can also find a missing part of the equation by using our techniques for solving equations.

Example 2.

What principal amount will grow to $3000 if invested at 6.5% compounded weekly for 4 years?

Identify information given:

$A = 3000$, $r = 0.065$, $n = 52$, $t = 4$

$3000 = P\left(1 + \frac{0.065}{52}\right)^{52\cdot 4}$ Evaluate parentheses

$3000 = P(1.00125)^{52\cdot 4}$ Multiply exponents

$3000 = P(1.00125)^{208}$ Evaluate exponent

$\frac{3000}{1.296719528\ldots} = \frac{P(1.296719528\ldots)}{1.296719528\ldots}$ Divide each side by $1.296719528\ldots$

$2313.53 = P$ Solution for P

$2313.53 Our Answer

It is interesting to compare the same investments made at several different types of compounding periods. The next few examples do just that.

Example 3.

If \$4000 is invested in an account paying 3% interest compounded monthly, what is the balance after 7 years?

Identify information given:

$P = 4000$, $r = 0.03$, $n = 12$, $t = 7$

$A = 4000\left(1 + \frac{0.03}{12}\right)^{12 \cdot 7}$ Plug each value into formula, evaluate parentheses

$A = 4000(1.0025)^{12 \cdot 7}$ Multiply exponents

$A = 4000(1.0025)^{84}$ Evaluate exponent

$A = 4000(1.2333548)$ Multiply

$A = 4933.42$

\$4933.42 Our Answer

To investigate what happens to the balance if the compounding happens more often, we will consider the same problem but with interest compounded daily.

Example 4.

If \$4000 is invested in an account paying 3% interest compounded daily, what is the balance after 7 years?

Identify information given:

$P = 4000$, $r = 0.03$, $n = 365$, $t = 7$

$A = 4000\left(1 + \frac{0.03}{365}\right)^{365 \cdot 7}$ Plug each value into formula, evaluate parentheses

$A = 4000(1.00008219\ldots)^{365 \cdot 7}$ Multiply exponent

$A = 4000(1.00008219\ldots)^{2555}$ Evaluate exponent

$A = 4000(1.23366741\ldots)$ Multiply

$A = 4934.67$

\$4934.67 Our Answer

While this difference in amounts in Examples 3 and 4 is not very large, it is still a bit higher when the interest is compounded more often. The table below shows the result for the same problem but with different compounding periods.

Compounding	Balance
Annually	$4919.50
Semi-Annually	$4927.02
Quarterly	$4930.85
Monthly	$4933.42
Weekly	$4934.41
Daily	$4934.67

As the table illustrates, the more often interest is compounded, the higher the final balance because we are calculating interest on interest. So once interest is added into the account, it will start earning interest for the next compounding period and thus giving a higher final balance.

COUNTINUOUSLY COMPOUNDED INTEREST

The next question one might consider is what is the maximum number of compounding periods possible? We actually have a way to calculate interest compounded an infinite number of times a year. This is when the interest is compounded *continuously*. When we see the word continuously we will know that we cannot use the first formula. Instead we will use the following formula:

CONTINUOUSLY COMPOUNDED INTEREST FORMULA

$$A = Pe^{rt}$$

A = final amount
P = principal (starting balance)
e = a constant approximately equal to 2.71828183…
r = annual interest rate (as a decimal)
t = time (in years)

The number e is a constant similar in idea to the number π in that it goes on forever without repeat or pattern. Just as π naturally occurs in several geometry applications, e appears in many exponential applications, continuous interest being one of them. If you have a scientific calculator you probably have an e^x button (often using the 2nd or shift key, then hit the ln button) that will be useful in calculating interest compounded continuously.

Example 5.

If $4000 is invested in an account paying 3% interest compounded continuously, what is the balance after 7 years?

	Identify information given:
	$P = 4000$, $r = 0.03$, $t = 7$
$A = 4000e^{0.03 \cdot 7}$	Plug each value into formula, multiply exponent
$A = 4000e^{0.21}$	Evaluate $e^{0.21}$
$A = 4000(1.23367806\ldots)$	Multiply
$A = 4934.71$	
$4934.71	Our Answer

Consider the following example, illustrating how powerful compound interest can be.

Example 6.

If you invest $6.16 in an account paying 12% interest compounded continuously for 100 years, and that is all you have to leave your children as an inheritance, what will the final balance be that they will receive?

	Identify information given:
	$P = 6.16$, $r = 0.12$, $t = 100$
$A = 6.16e^{0.12 \cdot 100}$	Plug each value into formula, multiply exponent
$A = 6.16e^{12}$	Evaluate e^{12}
$A = 6.16(162,754.79)$	Multiply
$A = 1,002,569.52$	
$1,002,569.52	Our Answer

In 100 years that one time investment of $6.16 investment grew to over one million dollars. That's the power of compound interest!

Practice Exercises
Section 5.5: Compound Interest

Find the balance when:

1) \$500 is invested at 4% compounded annually for 10 years.
2) \$600 is invested at 6% compounded annually for 6 years.
3) \$750 is invested at 3% compounded continuously for 8 years.
4) \$1500 is invested at 4% compounded semiannually for 7 years.
5) \$900 is invested at 6% compounded monthly for 5 years.
6) \$950 is invested at 4% compounded continuously for 12 years.
7) \$2000 is invested at 5% compounded quarterly for 6 years.
8) \$2250 is invested at 4% compounded daily for 9 years.
9) \$3500 is invested at 6% compounded continuously for 12 years.

Answer each question.

10) What principal will amount to \$2000 if invested at 4% interest compounded semiannually for 5 years?

11) What principal will amount to \$3500 if invested at 4% interest compounded quarterly for 5 years?

12) What principal will amount to \$3000 if invested at 3% interest compounded semiannually for 10 years?

13) What principal will amount to \$2500 if invested at 5% interest compounded semiannually for 7.5 years?

14) What principal will amount to \$1750 if invested at 3% interest compounded quarterly for 5 years?

15) A thousand dollars is left in a bank savings account drawing 7% interest, compounded quarterly for 10 years. What is the balance at the end of that time?

The Practice Exercises are continued on the next page.

Practice Exercises: Section 5.5 (continued)

Answer each question.

16) A thousand dollars is left in a credit union drawing 7% compounded monthly. What is the balance at the end of 10 years?

17) $1750 is invested in an account earning 13.5% interest compounded monthly for a 2 year period. What is the balance at the end of 2 years?

18) You lend out $5500 at 10% compounded monthly. If the debt is repaid in 18 months, what is the total owed at the time of repayment?

19) You borrow $25000 at 12.25% interest compounded monthly. If you are unable to make any payments the first year, how much do you owe, excluding penalties?

20) An 8.5% account earns continuous interest. If $2500 is deposited for 5 years, what is the total accumulated?

21) You lend $100 at 10% continuous interest. If you are repaid 2 months later, what is owed?

ANSWERS to Practice Exercises
Section 5.5: Compound Interest

1) $ 740.12

2) $ 851.11

3) $ 953.44

4) $ 1979.22

5) $ 1213.97

6) $ 1535.27

7) $ 2694.70

8) $ 3224.93

9) $ 7190.52

10) $ 1640.70

11) $ 2868.41

12) $ 2227.41

13) $ 1726.16

14) $ 1507.08

15) $ 2001.60

ANSWERS to Practice Exercises: Section 5.5 (continued)

16) $ 2009.66

17) $ 2288.98

18) $ 6386.12

19) $ 28240.43

20) $ 3823.98

21) $ 101.68