4.2 Linear Programming – The Graphical Method

Linear programming gained prominence in World War II with a variety of problems, including fuel-related problems. Gasoline and oil were precious commodities during war. Conserving these resources is a priority. For example, suppose we have 20 munitions storage locations and 30 requests for munitions from different locations. One possible linear programming problem is this: how do we fill all 30 of the requests (all deliveries are by truck) while at the same time using a minimum amount of fuel?

The problem above is long and involved. In fact, its solution was a factor leading to the development of the computer. We illustrate linear programming problems in detail with a simpler example.

Example 4.2.1: Shipping Cargo

A truck traveling from California to Oregon is to be loaded with two types of cargo. Each crate of cargo P is 4 cubic feet in volume, weighs 100 pounds, and earns $12 for the driver. Each crate of cargo Q is 3 cubic feet in volume, weighs 200 pounds, and earns $11 for the driver. The truck can carry no more than 300 cubic feet of crates, and no more than 10,000 pounds of cargo. In addition, the number of crates of cargo Q must be less than or equal to twice the number of crates of cargo P. How many crates of each type of cargo should the driver haul to maximize earnings?

Solution:

First, summarize the given information in a table. Include variables and totals in the table.

Type of Crate	Volume	Weight	Earnings Per Crate	Number of Crates
P	4 cubic feet	100 pounds	$12	x
Q	3 cubic feet	200 pounds	$11	y
Capacity	≤ 300 cubic feet	≤ 10,000 pounds		

The last sentence of the problems tells us how to define our variables by telling us what to find. Let x represent the number of crates of cargo P and y represent the number of crates of cargo Q.

A **constraint** is a condition imposed on a problem. Constraints in linear programming problems are stated as inequalities. The constraint in the second to last sentence is turned into an inequality by writing and translating:

number of crates of cargo Q ≤ twice the number of crates of cargo P

$$y \leq 2 \cdot x$$

Second, build the constraints and the objective function. A constraint is a condition imposed upon the variables in a problem. It is expressed as an inequality, using the symbols <, ≤, >, or ≥. In a table such as the previous, either the rows or columns (not both) will give us some of the constraints. The constraints for this problem are:

volume constraint	$4x + 3y \le 300$
weight constraint	$100x + 200y \le 10{,}000$
crates constraint	$y \le 2x$
nonnegative variables constraint	$x \ge 0, y \ge 0$

Each constraint reads as a condition. For example, look at the volume constraint. Since one crate of cargo P occupies 4 cubic feet of space in the truck (two crates occupy $2 \cdot 4 = 8$cubic feet, three crates occupy $3 \cdot 4 = 12$ cubic feet, and so on), then *x* crates occupy $x \cdot 4 = 4x$ cubic feet of space. Similarly, since each crate of cargo Q occupies 3 cubic feet of space, *y* crates occupy $3y$ cubic feet of space in the truck. Since the truck cannot fit any more than 300 cubic feet of cargo, we have the condition $4x + 3y \le 300$.

(*Aside*: Hence, you see a truism in mathematics. Mathematics reads just like English, but it is written in symbols. All of mathematics is this way. And this is wherein its challenge lies. Math is shorthand and the challenge is to get used to deciphering it.)

The conditions $x \ge 0$ and $y \ge 0$ are standard in linear programming applications. They are common sense. Try translating them into meaningful English sentences before reading the next sentence. Since *x* and *y* represent the number of crates of cargo, and we cannot load a negative number of crates into the truck, then *x* and *y* must be 0 (loads no crates into the truck) or positive. These conditions will make your linear programming life easier because they restrict the graph we draw to the first quadrant.

The earnings column in the table <u>is not</u> a constraint. Since the last sentence in the problem says to maximize earnings, this column gives us an equation. This earnings equation is called our objective function. We write it as:

$$E = \$12x + \$11y$$

Our objective is to find the number of crates of cargo *x* and *y* that maximize the trucker's earnings *E* subject to the five constraints.

Third, graph all of the constraints to find the feasible solution set. This step is the most work. I want to find all ordered pairs (x, y) that satisfy all five of the constraints (conditions) in this problem. There are an infinite number of such ordered pairs, represented by a shaded region on the graph. Then we want to sift through this infinity of points and find the single ordered pair that maximizes the earnings. This sounds like finding a needle in a haystack. Fret not! Linear programming theory provides us a shortcut. Let's begin to graph.

Graph the constraints $x \geq 0$and $y \geq 0$: These two constrains restrict our graph to the first quadrant. So begin by drawing the first quadrants. We will graph the other three constraints by finding the *x*- and *y*-intercepts, as done in the previous section. See the graph.

Graph $4x + 3y \leq 300$: We first graph the equation $4x + 3y = 300$. Find two points satisfying the equation. Points with 0 coordinates are easiest to work with. We called this the intercept-method in the previous section. We put our results in a small table of values below.

x	y	(x, y)
0	$4(0) + 3y = 300$ $3y = 300$ $y = 100$	(0,100)
$4x + 3(0) = 300$ $4x = 300$ $x = 75$	0	(75,0)

Plot the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $4x + 3y \leq 300$ (the "equals" in the symbol ≤ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point (0,0) because it is easy to work with: $4(0) + 3(0) \leq 300$. This results in $0 \leq 300$, which is a true statement. This means that every point on the same side of the line as (0,0) satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the southwestern part of the graph (Note: Write the equation of the line on its graph. We will label each line with the equation and use this labeling later.)

Graph $100x + 200y \leq 10{,}000$: Now we graph the equation $100x + 200y = 10{,}000$ using the same values for *x* and *y*.

x	y	(x, y)
0	$100(0) + 200y = 10{,}000$ $200y = 10{,}000$ $y = 50$	(0,50)
$100x + 200(0) = 10{,}000$ $100x = 10{,}000$ $x = 100$	0	(100,0)

Plot the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $100x + 200y \leq 10{,}000$ (again, the "equals" in the symbol ≤ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point (0,0) again because it is easy to work with: $100(0) + 200(0) \leq 10{,}000$. This results in $0 \leq 10{,}000$, which is a true statement. This means that every point on the same side of the line as (0,0) satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the southwestern part of the graph. Don't forget to write the equation of the line on its graph.

Graph $y \leq 2x$: Now we graph the equation $y = 2x$ using the same values for *x* and *y*.

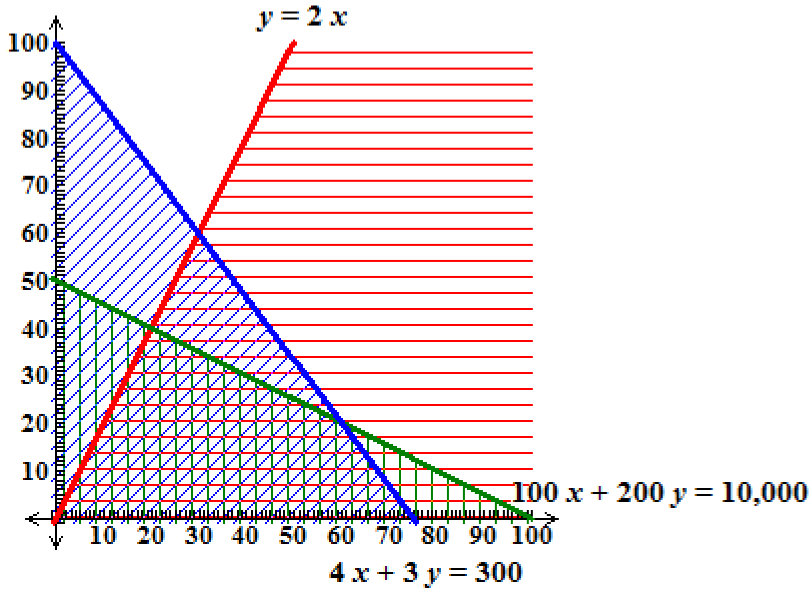
x	y	(x, y)
0	$y = 2(0)$ $y = 0$	$(0,0)$
$0 = 2x$ $0 = x$	0	$(0,0)$

Notice the *x*- and *y*-intercept are the same, namely the origin. Therefore, we need another point. Choose a random, reasonable number for *x* and solve for *y*.

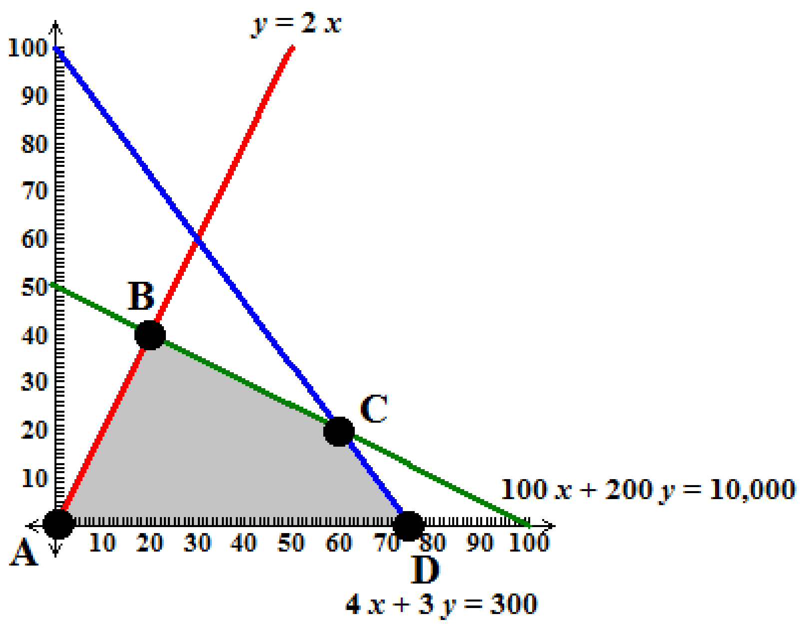
x	y	(x, y)
40	$y = 2(40)$ $y = 80$	$(40,80)$

Plot the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $y \leq 2x$ (again, the "equals" in the symbol $\leq$ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. We cannot test $(0,0)$ since it is on the line, so let's test $(0,10)$: $10 \leq 2(0)$. This results in $10 \leq 0$, which is a false statement. This means that every point on the same side of the line as $(0,10)$ does not satisfy the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the northeastern part of the graph. Don't forget to write the equation of the line on its graph.

When constructing the graph, you may wish to use colored pencils or a creative shading style. Ultimately, we want to find the region of the graph where all shaded areas overlap. On the graph, we used red horizontal lines when shading $y \leq 2x$, blue diagonal lines when shading $4x + 3y \leq 300$, and green vertical lines when shading $100x + 200y \leq 10{,}000$.



Now what?! The feasible solution set is the region all shaded areas overlap. It is the set of all points (x, y)that satisfy all five of our constraints simultaneously. We must find the single point (in the infinite number of points in the feasible solution set) that gives the trucker the greatest earnings. Fortunately for us, the **Fundamental Principle of Linear Programming** tells us that this point lies on the boundary of the feasible set. In fact, it is one of the four **vertex points**, also called **corner points**, labeled A, B, C, and D on the graph:



In other words, the **Fundamental Principle of Linear Programming** states that the ordered pair that maximizes or minimizes the objective function is a vertex on the boundary of the feasible set. The reason we graph the feasible set is to get the vertex points. Now we need to find the ordered pairs, (x, y), of vertices A, B, C, and D. Vertices A and D are easy to identify based on the graph and are $(0,0)$ and $(75,0)$, respectively. Vertices B and C must be determined by solving a system of equations. Don't attempt to guess the points based on the graph.

We want to find the intersection point of two lines. We will use a combination of the substitution and addition/elimination methods (A review of the substitution method can be found here and a review of the addition/elimination method can be found here).

Determining Vertex B: Vertex B is located at the intersection of the lines $y = 2x$ and $100x + 200y = 10{,}000$. Using the substitution method, we can substitute $2x$ in the place of y in the second equation:

$$\begin{gathered} 100x + 200(2x) = 10{,}000 \\ 100x + 400x = 10{,}000 \\ 500x = 10{,}000 \\ x = 20 \end{gathered}$$

If $x = 20$, substitute this for x into either equation:

$$\begin{gathered} y = 2x \\ y = 2(20) \\ y = 40 \end{gathered}$$

Therefore, vertex B is located at (20,40).

Determining Vertex C: Vertex C is located at the intersection of the lines $100x + 200y = 10{,}000$ and $4x + 3y = 300$. It may be more convenient to solve this system using the addition/elimination method. With this method, we want the coefficients of x or y to be opposites. One way to achieve this is to multiply each term of the second equation by -25. This gives the following system of equations.

$$\begin{cases} 100x + 200y = 10{,}000 \\ -100x - 75y = -7{,}500 \end{cases}$$

We now add these two equations together, term by term, to get $125y = 2{,}500$ or $y = 20$. Substitute $y = 20$ into either of the two original equations to solve for x:

$$\begin{aligned} &4x + 3(20) = 300 \\ &4x + 60 = 300 \\ &4x = 240 \\ &x = 60 \end{aligned}$$

Therefore, vertex C is located at (60,20). Now we determine which of the four vertices optimizes (in this case, maximizes) the objective function $E = 12x + 11y$.

Vertex	$E = 12x + 11y$
A (0,0)	$E = 12(0) + 11(0) = \$0$
B (20,40)	$E = 12(20) + 11(40) = 240 + 440$ $= \$680$
C (60,20)	$E = 12(60) + 11(20) = 720 + 220$ $= \$940$
D (75,0)	$E = 12(75) + 11(0) = 900 + 0 = \$900$

Recall that *x* represents the number of crates P to load onto the truck and *y* represents the number of crates Q. Based on the table, vertex C satisfies all five of the constraints and gives us maximum earnings. In conclusion, the truck driver will make a maximum earning of $940 if 60 crates of cargo P and 20 crates of cargo Q are hauled.

Try it Now 4.2.1

The AB Lacrosse Company manufactures two types of lacrosse sticks. Stick P requires 2 labor-hours for cutting, 1 labor-hour for stringing, and 2 labor-hours for finishing, and is sold for a profit of $12. Stick Q requires 1 labor-hour for cutting, 2 labor-hours for stringing, and 2 labor-hours for finishing, and is sold for a profit of $15. Each week the company has available 660 labor-hours for cutting, 680 labor-hours for stringing, and 750 labor-hours for finishing. How many lacrosse sticks of each type should be manufactured each week to maximize profits?

Example 4.2.2

Let's examine another example, where the linear programming model is already developed.

Minimize $C = 3x + 9y$ subject to

$$x + y \geq 35$$
$$2x + 5y \geq 100$$
$$x \geq 0$$
$$y \geq 0.$$

Solution:

Graph the constraints $x \geq 0$ and $y \geq 0$: These two constraints restrict our graph to the first quadrant. So begin by drawing the first quadrants. We will graph the other three constraints by finding the *x*- and *y*-intercepts, as done in the previous problem. See the graph.

Graph $x + y \geq 35$: We first graph the equation $x + y = 35$. Find two points satisfying the equation. Points with 0 coordinates are easiest to work with. We called this the intercept-method in the previous section. We put our results in a small table of values.

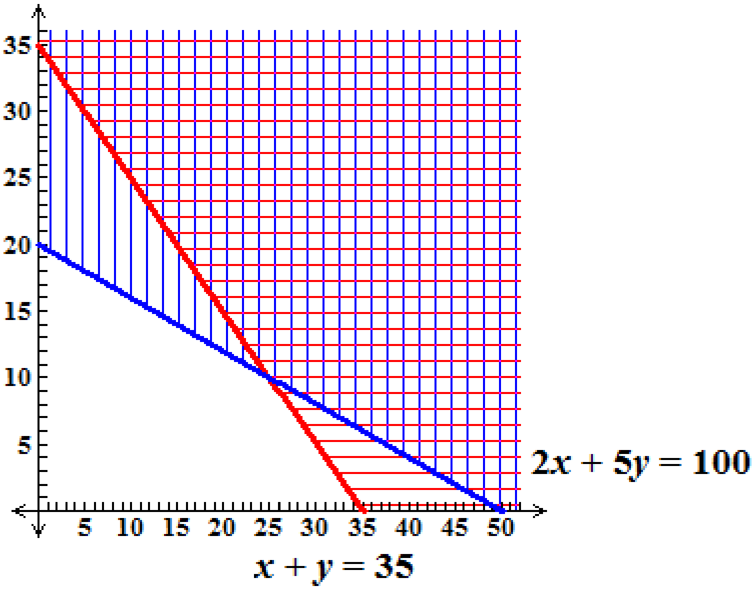
$\boldsymbol{x}$	$\boldsymbol{y}$	$(\boldsymbol{x},\boldsymbol{y})$
0	$(0) + y = 35$ $y = 35$	(0,35)
$x + (0) = 35$ $x = 35$	0	(35,0)

Plot the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $x + y \geq 35$ (the "equals" in the symbol ≥ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point $(0,0)$ because it is easy to work with: $0 + 0 \geq 35$. This results in $0 \geq 35$, which is a false statement. This means that every point on the other side of the line as $(0,0)$ satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the northeastern part of the graph. (Note: Write the equation of the line on its graph. We will label each line with the equation and use this labeling later.)

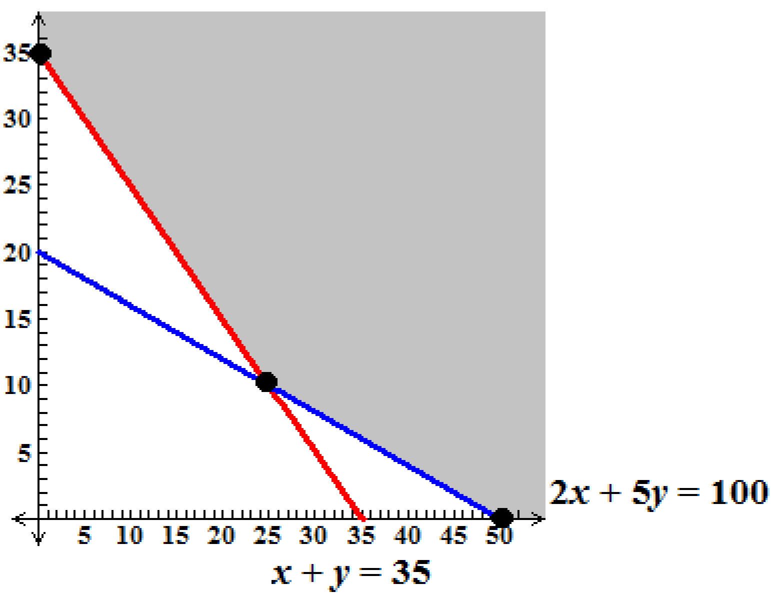
Graph $2x + 5y \geq 100$: Now we graph the equation $2x + 5y = 100$using the same values for *x* and *y*.

$\boldsymbol{x}$	$\boldsymbol{y}$	$(\boldsymbol{x}, \boldsymbol{y})$
0	$2(0) + 5y = 100$ $5y = 100$ $y = 20$	$(0,20)$
$2x + 5(0) = 100$ $2x = 100$ $x = 50$	0	$(50,0)$

Plot the points and connect them with a straight line (see the graph). Now we are finished graphing the equation (the line). Return to the inequality. All of the points on the line are solutions to the inequality $2x + 5y \geq 100$ (again, the "equals" in the symbol ≥ guarantees this). To determine which side of the line, test any point not on the line by substituting its coordinates for *x* and *y* in the inequality. Suppose we test the point $(0,0)$ again because it is easy to work with: $2(0) + 5(0) \geq 100$. This results in $0 \geq 100$, which is a false statement. This means that every point on the other side of the line as $(0,0)$ satisfies the inequality. We want to shade the side of the line that represents the solution set. Therefore, we shade the northeastern part of the graph. Don't forget to write the equation of the line on its graph.



When constructing the graph, you may wish to use colored pencils or a creative shading style. Ultimately, we want to find the region of the graph where all shaded areas overlap. On the graph, we used red horizontal lines when shading $x + y \geq 35$ and blue vertical lines when shading $2x + 5y \geq 100$. The area where all regions overlap is the northeastern part of the graph and this establishes three vertices.



We have three vertices. Two vertices are obvious: $(0,35)$ and $(50,0)$. The third is located at the intersection of the lines $x + y = 35$ and $2x + 5y = 100$. It may be more convenient to solve this system using the addition/elimination method. With this method, we want the coefficients of *x* or *y* to be opposites. One way to achieve this is to multiple each term of the first equation by -2. This gives the following system of equations.

$$\begin{cases} -2x - 2y = -70 \\ 2x + 5y = 100 \end{cases}$$

We now add these two equations together, term by term, to get $3y = 30$or $y = 10$. Plug $y = 10$ into either of the two original equations to solve for x:

$$x + y = 35$$
$$x + (10) = 35$$
$$x = 25$$

Therefore, the third vertex is located at $(25,10)$. Now we determine which of the three vertices optimizes (in this case, minimizes) the objective function $C = 3x + 9y$.

Vertex	$C = 3x + 9y$
$(0,35)$	$C = 3(0) + 9(35) = 315$
$(50,0)$	$C = 3(50) + 9(0) = 150$
$(25,10)$	$C = 3(25) + 9(10) = 165$

The minimum value, $C = 150$, is attained when $x = 50$ and $y = 0$.

Try it Now 4.2.2

Minimize $\boldsymbol{W = 8x + 4y}$ subject to

$$\boldsymbol{x + 2y \geq 56}$$
$$\boldsymbol{3x + 4y \geq 120}$$
$$\boldsymbol{x \geq 0}$$
$$\boldsymbol{y \geq 0}$$

Although this method tests your graphing skills, it is efficient for solving simpler linear programming models in two-variables. This method is no longer practical when working with many equations or with equations in three or more variables. In the next section, we will examine a non-graphical approach for deriving solutions to a linear programming model.

Try it Now Answers

4.2.1 The maximum profit of $5,415 is attained when 70 units of stick P and 305 units of stick Q are produced.

4.2.2 The minimum value, $W = 120$, is attained when $x = 0$ and $y = 30$.

Section 4.2 Exercises

Graph the feasible region for each set of inequalities. Find the vertex points.

1. $x + 2y \leq 4$
   $2x - y \leq 3$
   $x \geq 0$
   $y \geq 0.$

2. $x - y \leq 3$
   $x + y \leq 5$
   $x \geq 0$
   $y \geq 0$

Solve each linear programming programing by graphing and then determining which vertex maximizes or minimizes the objective function.

3. Maximize $M = 4x + 2y$ subject to
   $2x + 3y \leq 24$
   $x + 6y \leq 30$
   $x \geq 0$
   $y \geq 0$

4. Minimize $P = 2x + 3y$ subject to
   $3x + y \geq 18$
   $x + 5y \geq 20$
   $x \geq 0$
   $y \geq 0$

5. Maximize $M = 7x + 9y$ subject to
   $x + 2y \leq 10$
   $4x + y \leq 12$
   $x \geq 0$
   $y \geq 0$

6. Minimize $C = 10x + 13y$ subject to
   $x + y \geq 13$
   $10x + 3y \geq 60$
   $x \geq 0$
   $y \geq 0$

Solve each application problem.

7. The Toys Galore Company makes baseball and football games. Its manufacturing process has the following labor constraints. Each baseball game requires 2 hours of assembly and 2 hours of testing. Each football game requires 3 hours of assembly and 1 hour of testing. Each day there are 48 hours available for assembly and 32 hours available for testing. How many of each game should Toys Galore make each day to maximize the total number of games produced daily?

8. Sunny Contractors builds two types of homes. The rancher requires one lot, $25,000 in capital, and 160 man-days to build, and is sold for a profit of $4,800. The colonial requires two lots, $40,000 in capital, and 200 man-days to build, and is sold for a profit of $5,800. The contractor owns 200 lots, and has $4,600,000 in capital and 28,000 man-days of labor. If the company is certain all homes will sell, how many homes of each type should Sunny Contractors build in order to maximize its profit?

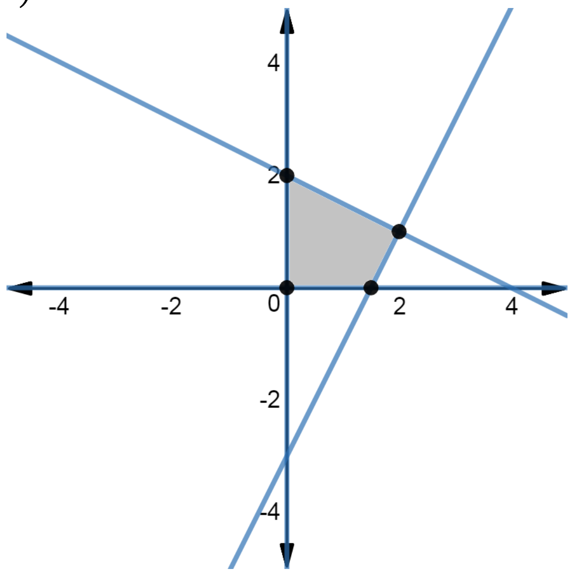
9. Coal Inc. owns two mines. On each day of the week the Pennsylvania mine produces 8 tons of anthracite (hard coal), 10 tons of bituminous (semi-hard coal), and 12 tons of lignite (soft coal). On each day of the week the Ohio mine produces 12 tons of anthracite, 8 tons of bituminous, and 6 tons of lignite. It costs the company $3,000 per day to operate the Pennsylvania mine and $3,800 per day to operate the Ohio mine. Coal Inc. receives an order for 144 tons of anthracite, 152 tons of bituminous, and 120 tons of lignite. How many days do we operate each mine so that we fill the order while keeping the cost to a minimum?

10. A company produces music players in factories in Baltimore and Frederick. Each week the Baltimore factory produces 800 economy music players and 500 premium music players. Each week the Frederick factory produces 500 economy music players and 500 premium music players. The company receives an order from an electronics store for 30,000 of the economy music players and 24,000 of the premium music players. It costs the Baltimore factory $24,000 per week to operate and the Frederick factory $20,000 per week to operate. How many weeks should each factory operate to fill the store order at the least cost?

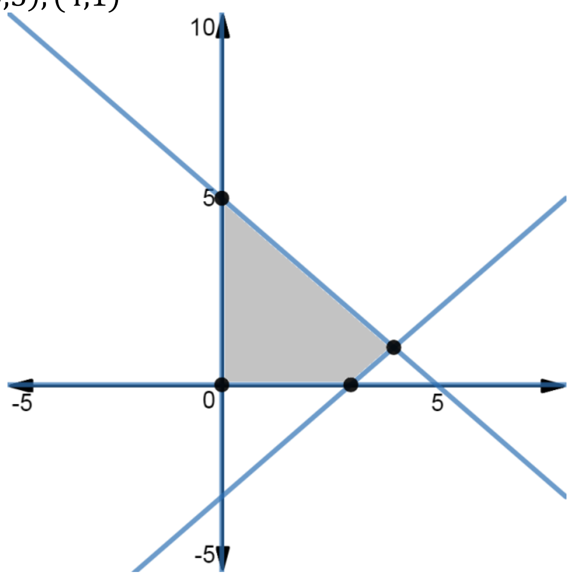
11. A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

Section 4.2 Exercises – Answer Key

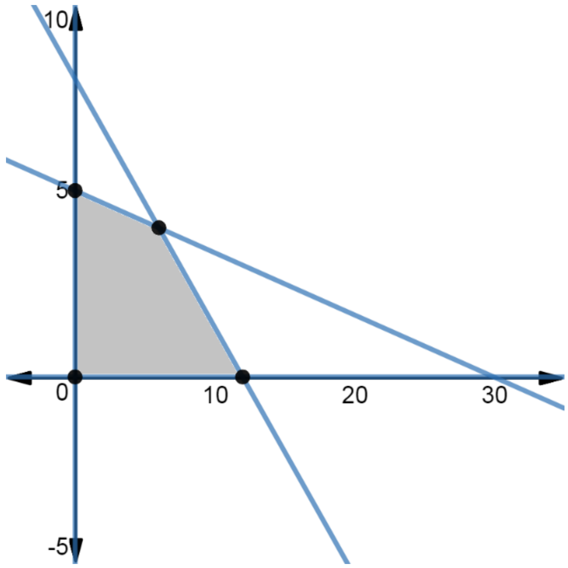
1. Vertices: $(0,0), \left(\frac{3}{2}, 0\right), (0,2), (2,1)$



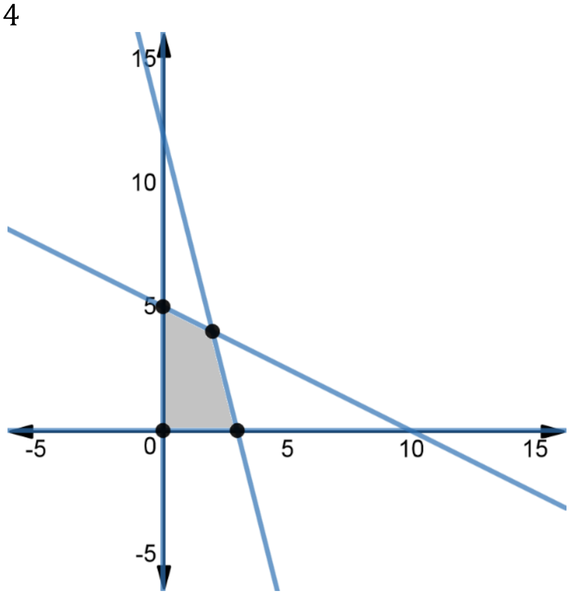
2. Vertices: $(0,0), (3,0), (0,5), (4,1)$



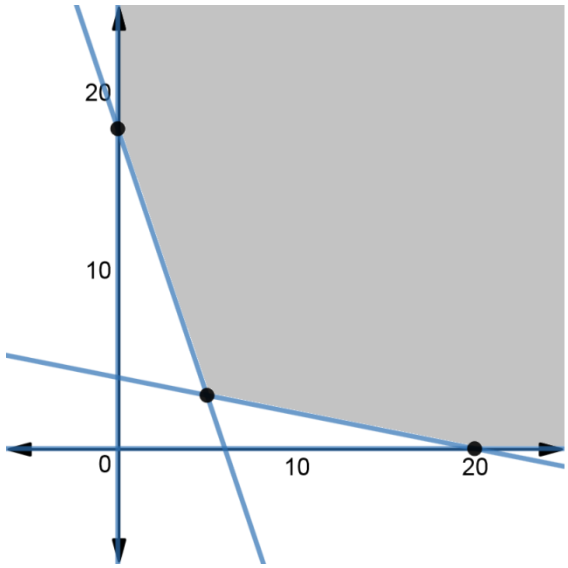
3. $M = 48, x = 12, y = 0$



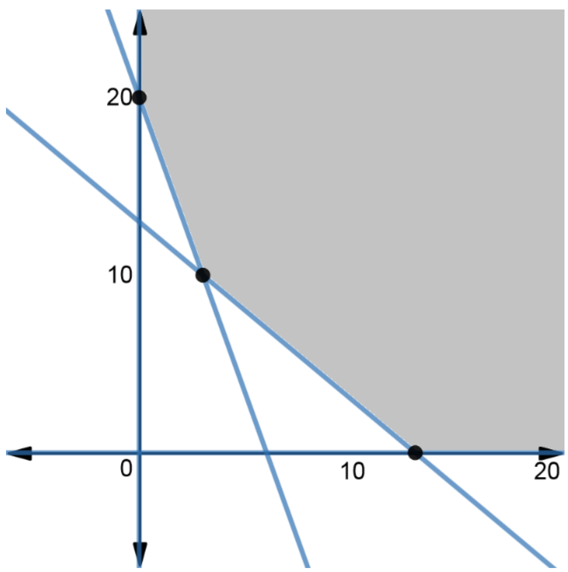
4. $M = 50, x = 2, y = 4$



5. $P = 19, x = 5, y = 3$



6. $C = 130, x = 13, y = 0$



7. Maximum production of 20 games is attained when 12 baseball and 8 football games are produced.
8. Maximum profit of $840,000 is attained when 175 ranchers and 0 colonials are built.
9. Minimum cost of $51,200 is attained when the Pennsylvania mine operates for 12 days and the Ohio mine operates for 4 days.
10. Minimum cost of $1,040,000 is attained when the Baltimore factory operates 20 weeks and the Frederick factory operates 28 weeks.
11. Maximum profit of $8,400 is attained when 80 acres of wheat and 20 acres of corn are planted.