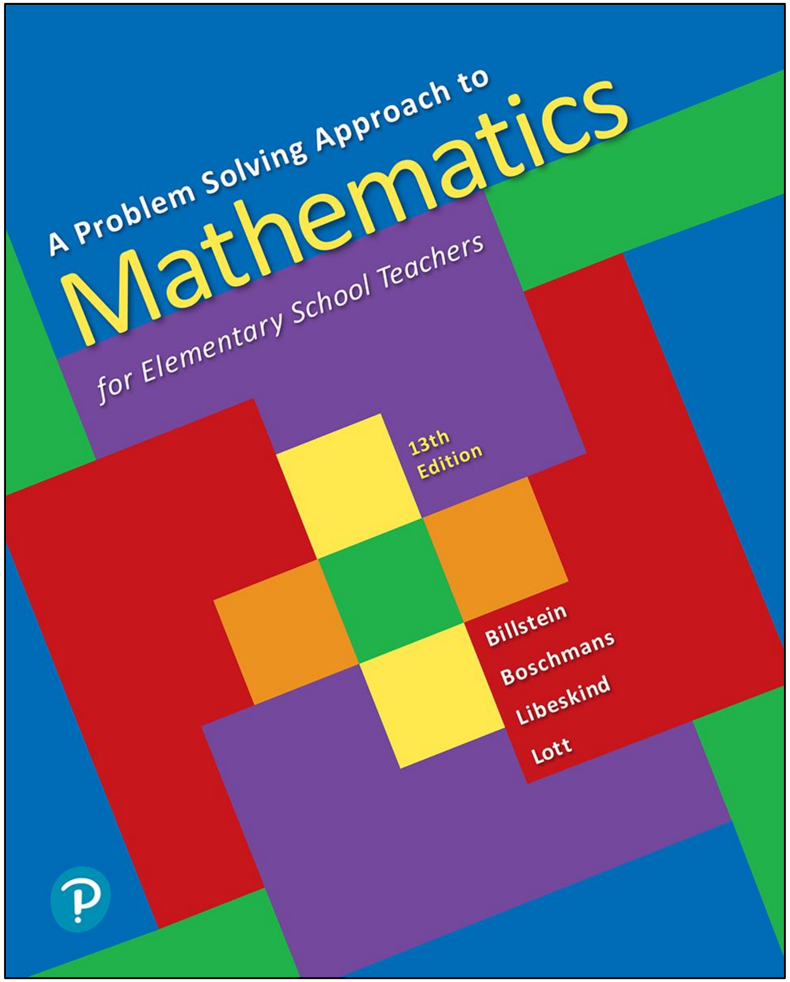
A Problem Solving Approach to Mathematics for Elementary School Teachers

Thirteenth Edition



Chapter 1

An Introduction to Problem Solving

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Section 1-1 Mathematics and Problem Solving

Students will be able to understand and explain

- The four-step problem solving process.
- How to solve problems using various problem solving strategies.

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Four-Step Problem-Solving Process

1. Understanding the problem.
2. Devising a plan.
3. Carrying out the plan.
4. Looking back.

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1. Understanding the Problem

a. Can the problem be stated differently?

b. What is to be found or what is needed?

c. What are the unknowns.

d. What information is obtained from the problem.

e. What information, if any, is missing or not needed?

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2. Devising a Plan (1 of 2)

- Look for a pattern.
- Examine a related problem.
- Examine a simpler or special case of the problem.
- Make a table or a list.
- Identify a subgoal.
- Make a diagram.

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2. Devisi a Plan (2 of 2)

- Use guess and check.
- Work backward.
- Write an equation.

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3. Carrying out the Plan

- Implement the strategy and perform any necessary actions or computations.
- Attend to precision in language and mathematics used.
- Check each step as you proceed.
- Keep an accurate record of your work.

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4. Looking Back

- Check the results in the original problem.
- Interpret the results in terms of the original problem.
- Does the answer make sense?
- Is it a reasonable answer?
- Determine if there is another method to find the solution.
- If possible, determine other related or more general problems for which the strategies will work.

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Strategies for Problem Solving

Strategies are tools that might be used to discover or construct the means to achieve a goal.

Because problems may be solved in more than one way, there is no one best strategy to use.

Sometimes strategies can be combined in order to solve a problem.

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Strategy: Look for a Pattern

Gauss's Problem

When Carl Gauss was a child, his teacher required the students to find the sum of the first 100 natural numbers. The teacher expected this problem to keep the class occupied for some time. Gauss gave the answer almost immediately. How did he solve the problem?

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Understanding the Problem 1

The natural numbers are $1, 2, 3, 4, \ldots$

The problem is to find the sum

$$1+2+3+4+\ldots+100$$

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Devising a Plan 1

List the numbers as shown below:

Let $S = 1 + 2 + 3 + 4 + \ldots + 100$. Then,

$$
\begin{array}{rcrrrrrrrr}
S & = & 1+ & 2+ & 3+ & 4+ & 5+\ldots+ & 98+ & 99+ & 100 \\
S & = & 100+ & 99+ & 98+ & 97+ & 96+\ldots+ & 3+ & 2+ & 1 \\
\hline
2S & = & 101+ & 101+ & 101+ & 101+ & 101+\ldots+ & 101+ & 101+ & 101
\end{array}
$$

Now divide the sum, $2S$, by 2.

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Carrying out the Plan 1

There are 100 sums of 101.

$$2S = 100 \cdot 101$$

$$S = \frac{100 \cdot 101}{2} = 5050$$

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Looking Back 1

The method is mathematically correct because addition can be performed in any order, and multiplication is repeated addition.

The sum in each pair is always 101 because when we move from any pair to the next, we add 1 to the top and subtract 1 from the bottom, which does not change the sum.

$$2+99=(1+1)+(100-1)=1+100,$$

$$3+98=(2+1)+(99-1)=2+99=101, \text{ etc.}$$

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Strategy: Examine a Related Problem

Find the sum of the even natural numbers less than or equal to 100. Devise a strategy for finding that sum and generalize the result.

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Understanding the Problem 2

Even natural numbers are $2, 4, 6, 8, 10, \ldots$ The problem is to find the sum of the even natural numbers.

$$2 + 4 + 6 + 8 + \ldots + 100$$

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Devising a Plan 2

Recognizing that the sum can be separated into two simpler parts related to Gauss's original problem helps us devise a plan.

$$\begin{aligned} 2+4+6+8+\ldots+100 &= 2\cdot 1+2\cdot 2+2\cdot 3+2\cdot 4+\ldots+2\cdot 50 \\ &= 2\left(1+2+3+4+\ldots+50\right) \end{aligned}$$

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Carrying out the Plan 2

$$2 + 4 + 6 + 8 + \ldots + 100 = 2(1 + 2 + 3 + 4 + \ldots + 50)$$

$$= 2\left(\frac{50 \cdot 51}{2}\right)$$

$$= 2550$$

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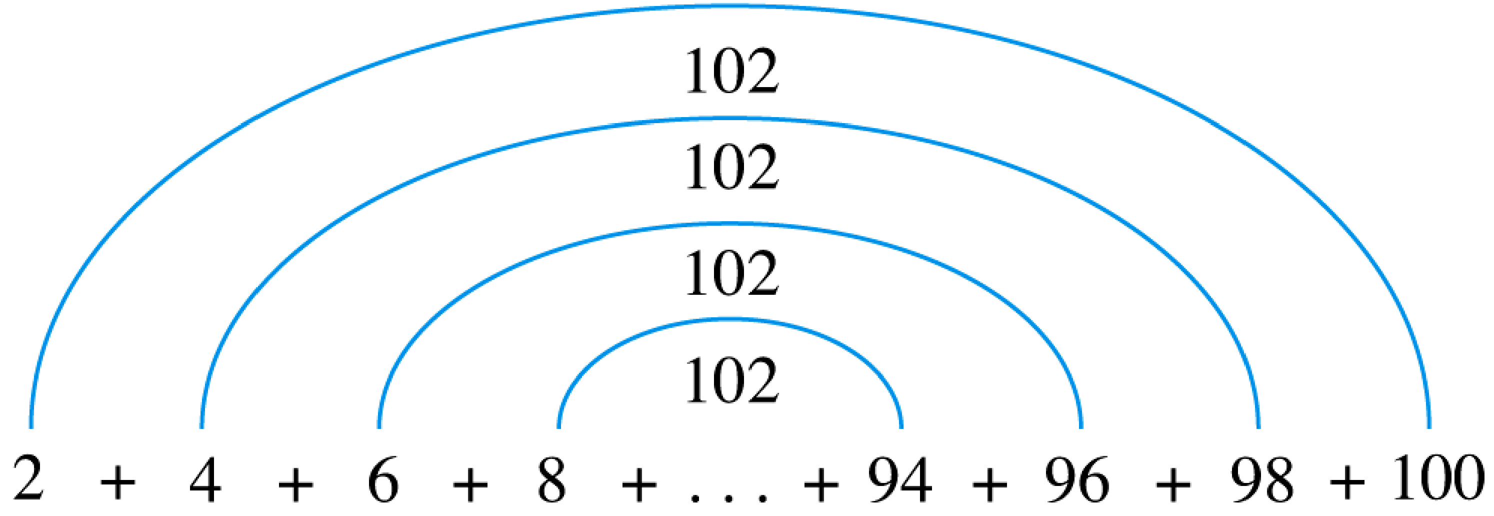
Looking Back 2

A different way to approach the problem is to realize that there are 25 sums of 102.



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Strategy: Examine a Simpler Case

One strategy for solving a complex problem is to examine a simpler case of the problem and then consider other parts of the complex problem.

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Strategy: Make a Table

An often-used strategy in elementary school mathematics is making a table

A table can be used to look for patterns that emerge in the problem, which in turn can lead to a solution.

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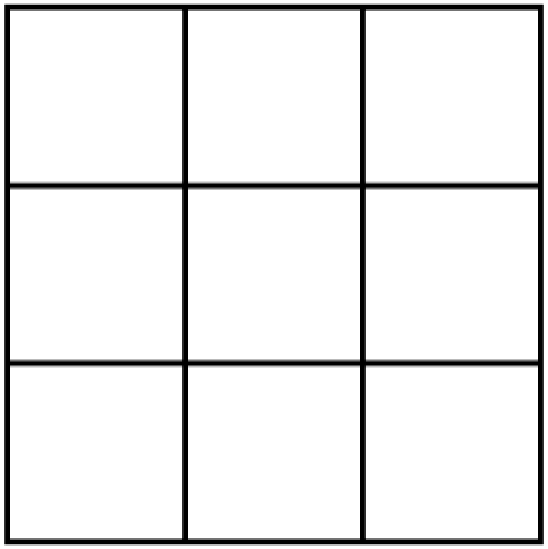
Strategy: Identify a Subgoal

Arrange the numbers 1 through 9 into a square subdivided into nine smaller squares like the one shown in the figure so that the sum of every row, column, and main diagonal is the same.



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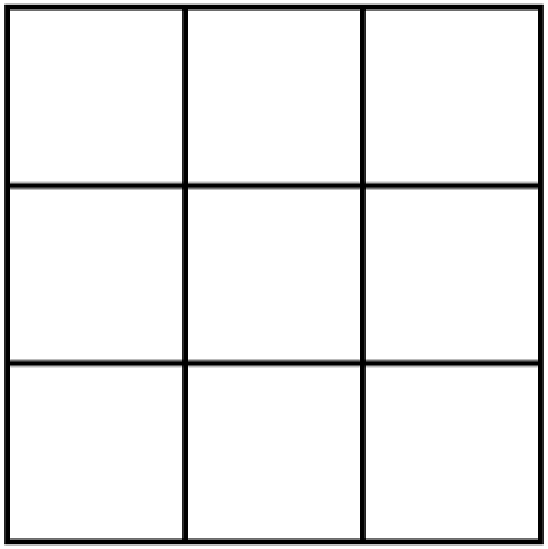
Understand the Problem 3

We need to put each of the nine numbers $1, 2, 3, \ldots, 9$ in the small squares, a different number in each square, so that the sums of the numbers in each row, in each column, and in each of the two major diagonals are the same.



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Devising a Plan 3 (1 of 3)

If we knew the fixed sum of the numbers in each row, column, and diagonal, we would have a better idea of which numbers can appear together in a single row, column, or diagonal.

Our **subgoal** is to find that fixed sum.

The sum of the nine numbers $1+2+3+\ldots+9$ equals three times the sum in one row.

The fixed sum is $\dfrac{1+2+3+\ldots+9}{3}=15.$

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Devising a Plan 3 (2 of 3)

Next, we need to decide what numbers could occupy the various squares.

The number in the center space will appear in four sums, each adding to 15 (two diagonals, the second row, and the second column).

Each number in the corners will appear in three sums of 15.

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Devising a Plan 3 (3 of 3)

If we write 15 as a sum of three different numbers 1 through 9 in all possible ways, we could then count how many sums contain each of the numbers 1 through 9.

The numbers that appear in at least four sums are candidates for placement in the center square, whereas the numbers that appear in at least three sums are candidates for the corner squares.

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Carrying out the Plan 3 (1 of 2)

The sums of 15 can be written as

$$9 + \textcircled{5} + 1$$
$$9 + 4 + 2$$
$$8 + 6 + 1$$
$$8 + \textcircled{5} + 2$$
$$8 + 4 + 3$$
$$7 + 6 + 2$$
$$7 + \textcircled{5} + 3$$
$$6 + \textcircled{5} + 4$$

The only number that appears in four sums is 5, so 5 must be in the center of the square.

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Carrying out the Plan 3 (2 of 2)

Because 2, 4, 6, and 8 appear 3 times each, they must go in the corners.

Suppose we choose 2 for the upper left corner. Then 8 must be in the lower right corner.

2	7	6
9	5	1
4	3	8

Now we could place 6 in the lower left corner or upper right corner and then complete the magic square.

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Looking Back 3

We have seen that 5 was the only number among the given numbers that could appear in the center.

However, we had various choices for a corner, and so it seems that the magic square we found is not the only one possible.

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Strategy: Make a Diagram

Bev and Jim ran a 50-m race three times. The speed of the runners did not vary. In the first race, Jim was at the 45-m mark when Bill crossed the finish line.

a. In the second race, Jim started 5 m ahead of Bev, who lined up at the starting line. Who won the race?

b. In the third race, Jim started at the starting line and Bev starts 5 m behind. Who will win the race?

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Understand the Problem 4

When Bev and Jim run a 50-m race, Bev wins by 5 m; that is, whenever Bev covers 50 m, at the same time Jim covers only 45 m.

If Bev starts at the starting line and Jim is given a 5-m head start, we are to determine who will win the race.

If Jim starts at the starting line and Bev starts 5 m behind, we are to determine who will win.

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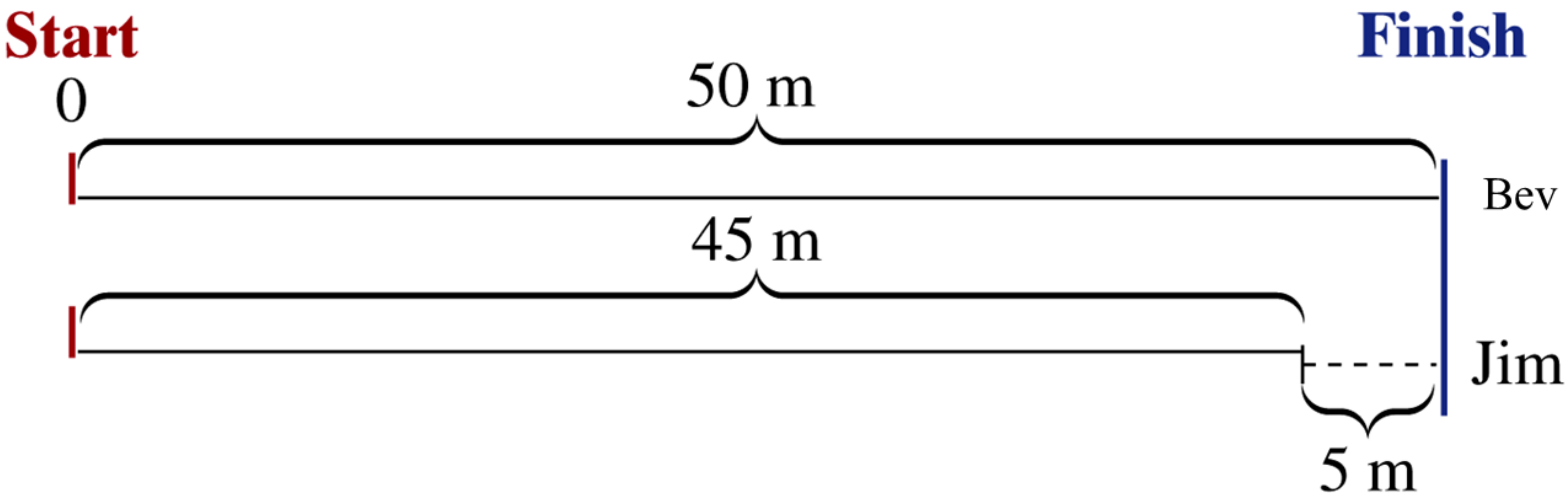
Devise a Plan 4 (1 of 3)

Race 1:



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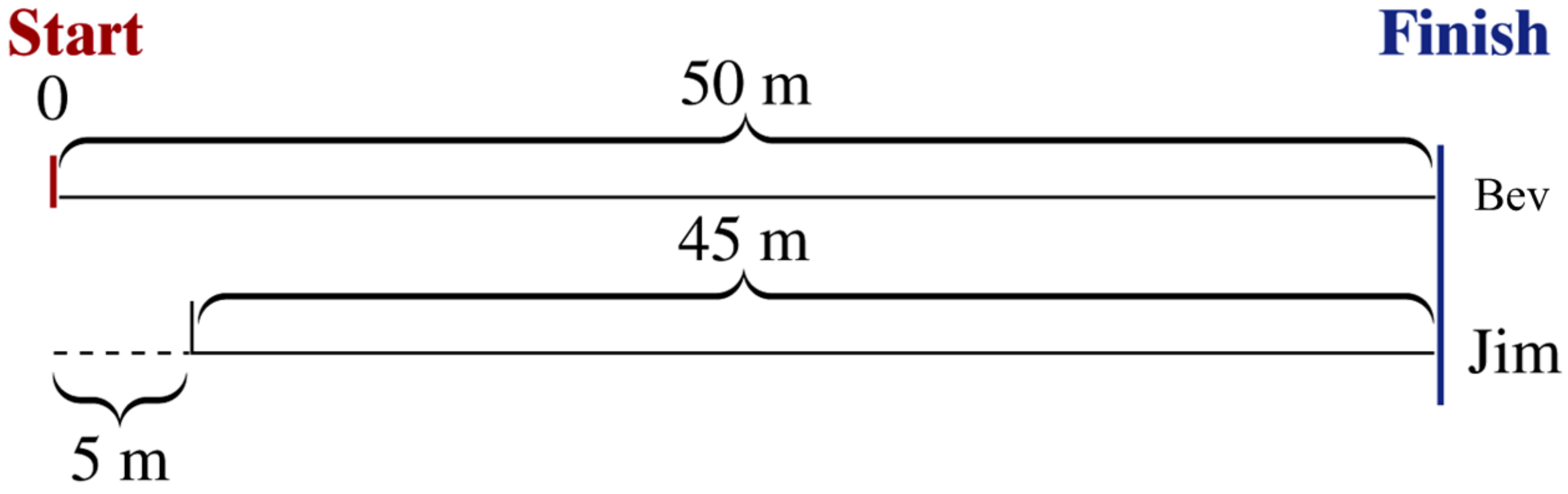
Devise a Plan 4 (2 of 3)

Race 2:



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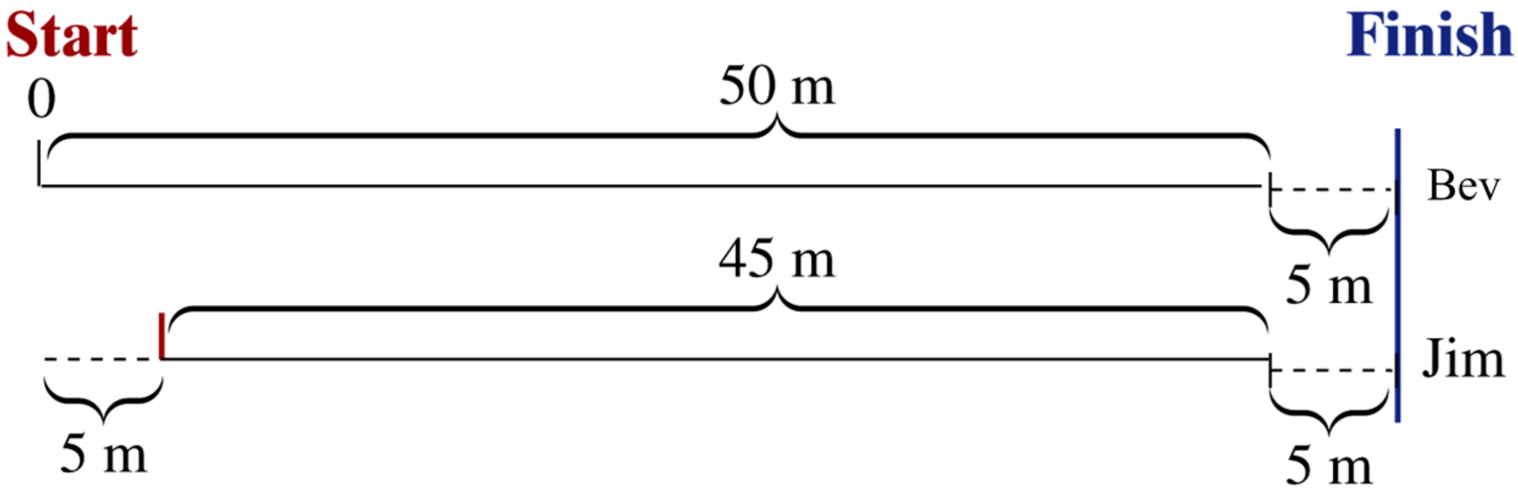
Devise a Plan 4 (3 of 3)

Race 3:



They will be tied at 45 m.

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Carry out the Plan 4

From the diagrams, we can determine the results in each case.

Race 1: Bev wins by 5 meters.

Race 2: They reach the finish line at the same time.

Race 3: Since Bev is faster than Jim, Bev will cover the last 5 m faster than Jim and will win the race.

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Look Back 4

The diagrams show the solution makes sense and is appropriate.

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Strategy: Work Backward

In some problems, it is easier to start with the result and to work backward.

It took workers 5 weeks to dig a 10-mile tunnel. During the fourth week, the workers dug $2\frac{1}{2}$ miles. The next week, they dug $1\frac{1}{4}$ miles to complete the tunnel. How much had the workers completed after the first 3 weeks of digging?

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Understand the Problem 5

The workers completed a 10-mile tunnel in 5 weeks. During week 4, they dug $2\frac{1}{2}$ miles, and during week 5, they dug $1\frac{1}{4}$ miles.

We need to find how many miles of tunnel they dug in the first 3 weeks.

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Devise a Plan 5

We know how many miles they dug in the last two weeks, so we will work backwards.

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Carry out the Plan 5

In weeks 4 and 5, they dug $2\frac{1}{2}+1\frac{1}{4}=3\frac{3}{4}$ miles.

So, in weeks 1 through 3, they dug

$10-3\frac{3}{4}=6\frac{1}{4}$ miles.

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Look Back 5

The answer makes sense because

$$6\frac{1}{4}+2\frac{1}{2}=1\frac{1}{4}=10 \text{ miles.}$$

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Strategy: Guess and Check

First guess at a solution using as reasonable a guess as possible. Then check to see whether the guess is correct. The next step is to learn as much as possible about the solution based on the guess before making a next guess.

This strategy is often used when a student does not have the tools to solve the problem in a faster way.

It is used primarily by students in grades 1–3.

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Strategy: Write an Equation

Even though algebraic thinking is involved in the strategy writing an equation and may evoke thoughts of traditional algebra, a closer look reveals that algebraic thinking starts very early in students' school lives. For example, finding the missing subtrahend in a problem like

$$\begin{array}{r} 14 \\ -\ \square \\ \hline 3 \end{array}$$

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