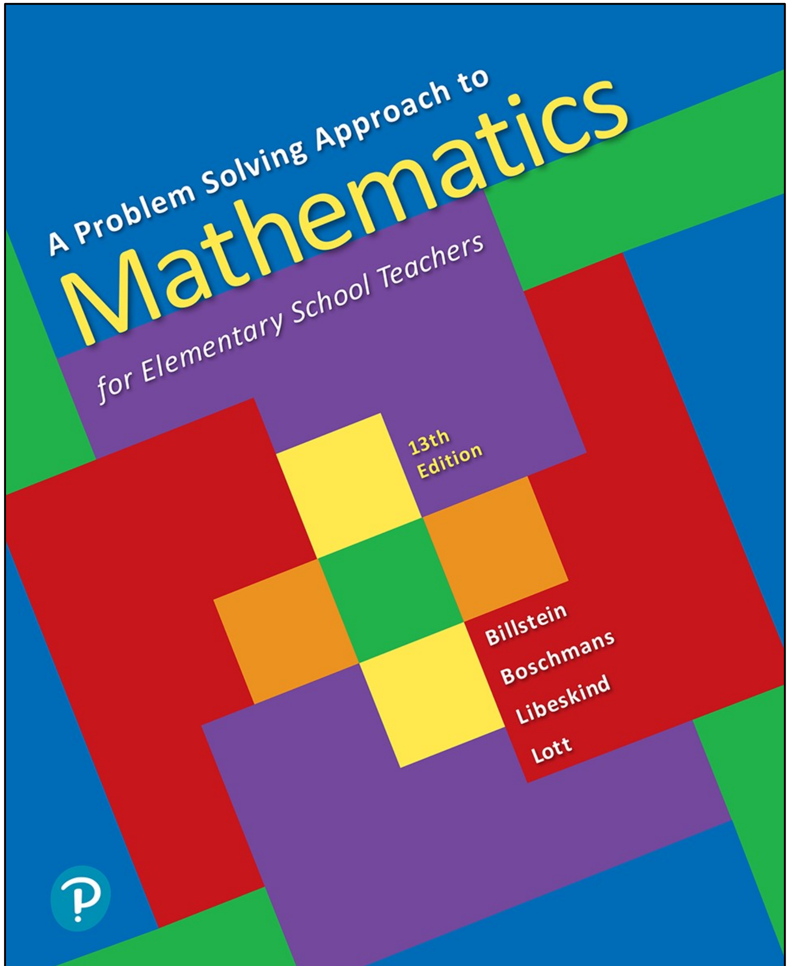
A Problem Solving Approach to Mathematics for Elementary School Teachers

Thirteenth Edition



Chapter 3

Numeration Systems and Whole Number Operations

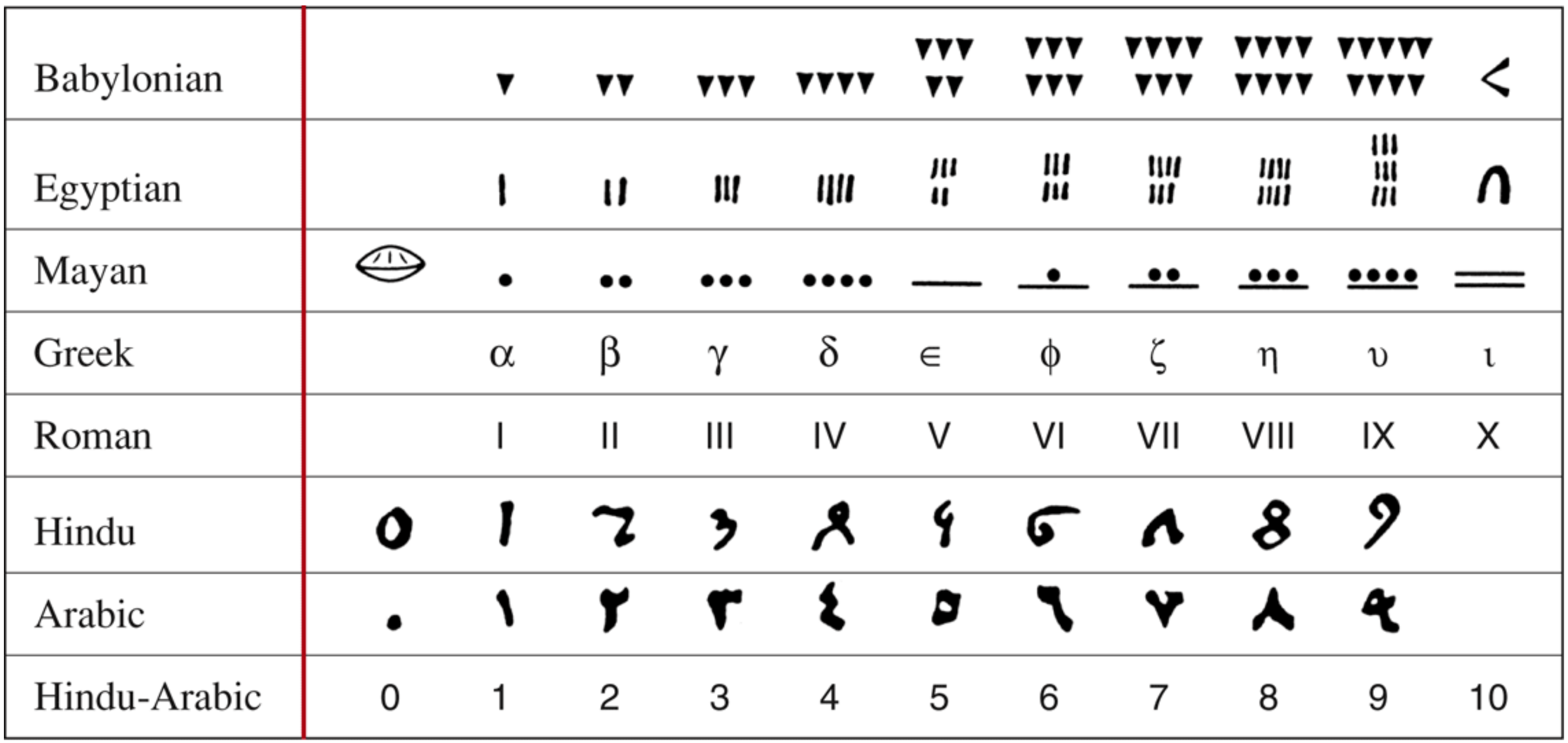
Section 3-1 Numeration Systems

Students will be able to understand and explain

- Numbers, their origin, and their representation in numerals and models.
- Different numeration systems including the Hindu-Arabic system.
- Place value and counting in base ten and other bases.

Definition (1 of 2)

Numerals: written symbols to represent cardinal numbers.

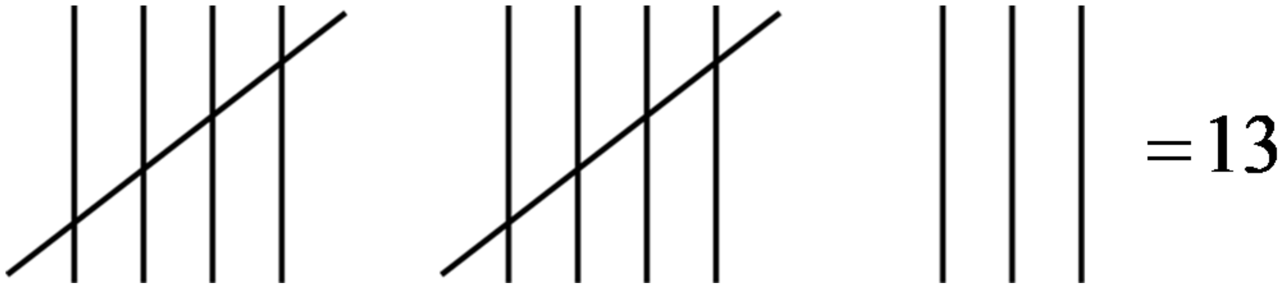


Definition (2 of 2)

Numeration system: a collection of properties and symbols agreed upon to represent numbers systematically.

Tally Numeration System

Uses single strokes (tally marks) to represent each object that is counted.



Egyptian Numeration System

Egyptian Numeral	Description	Hindu-Arabic Equivalent
	Vertical Staff	1
	Heel bone	10
	Scroll	100
	Lotus flower	1,000
	Pointing finger	10,000
	Polliwog or burbot	100,000
	Astonished man	1,000,000

Babylonian Numeration System

Babylonian Numeral	Hindu-Arabic Equivalent
𒁹	1
𒌋	10

Mayan Numeration System

Mayan Numeral	Hindu-Arabic Equivalent
•	1
—	5
	0

Roman Numeration System (1 of 2)

Roman Numeral	Hindu-Arabic Equivalent
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Roman Numeration System (2 of 2)

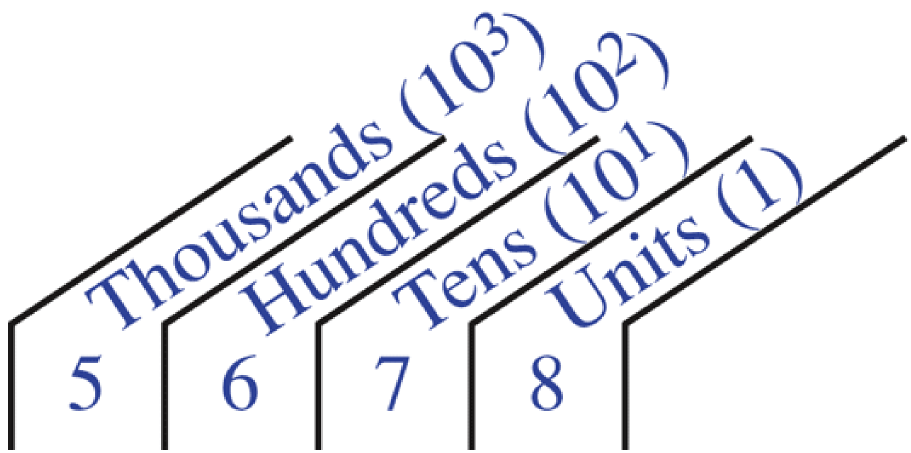
Roman Numeral	Hindu-Arabic Equivalent
IV	5 – 1, or 4
IX	10 – 1, or 9
XL	50 – 10, or 40
XC	100 – 10, or 90
CD	500 – 100, or 400
CM	1000 – 100, or 900

Hindu-Arabic Numeration System

1. All numerals are constructed from the 10 digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
2. Place value is based on powers of 10, the *number base* of the system.

Place Value

Place value assigns a value to a digit depending on its placement in a numeral.



Expanded form

$$6789 = 6 \times 10^3 + 7 \times 10^2 + 8 \times 10^1 + 9 \times 1$$

Definition of a^n

If a is any number and n is any natural number, then

$$a^n = \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}$$

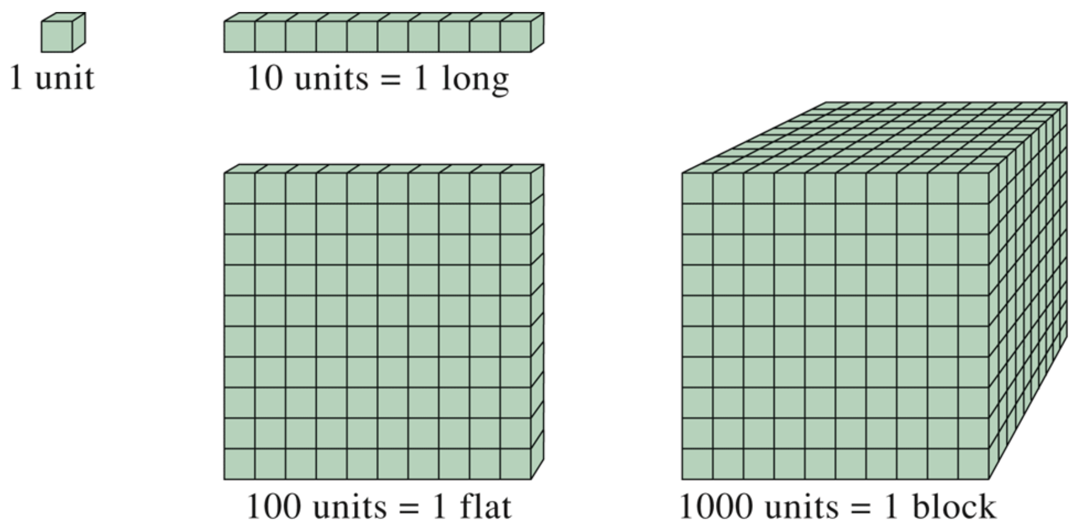
If $a \neq 0,$ then $a^0 = 1.$

Base-Ten Blocks

$1 \text{ long} \rightarrow 10^1 = 1$ row of 10 units

$1 \text{ flat} \rightarrow 10^2 = 1$ row of 10 longs, or 100 units

$1 \text{ block} \rightarrow 10^3 = 1$ row of 10 flats, or 100 longs, or 1000 units



Example 1 (1 of 3)

What is the fewest number of pieces you can receive in a fair exchange for 11 flats, 17 longs, and 16 units?

$$\begin{array}{lll} 11\text{ flats} & 17\text{ longs} & \cancel{16\text{ units}} \quad (16\text{ units} = 1\text{ long} \\ & 1\text{ long} & 6\text{ units} \quad \text{and } 6\text{ units}) \\ \hline 11\text{ flats} & 18\text{ longs} & 6\text{ units} \quad \text{(after the first trade)} \end{array}$$

Example 1 (2 of 3)

$$
\begin{array}{llll}
11\,\text{flats} & \cancel{18\,\text{longs}} & 6\,\text{units} & (18\,\text{longs} = 1\,\text{flat} \\
1\,\text{flat} & 8\,\text{longs} & & \text{and } 8\,\text{longs}) \\
\hline
12\,\text{flats} & 8\,\text{longs} & 6\,\text{units} & \text{(after the second trade)}
\end{array}
$$

Example 1 (3 of 3)

(12 flats = 1 block and 2 flats)

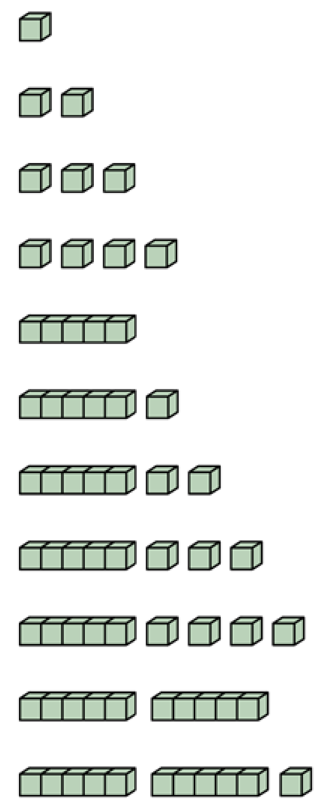
$$
\begin{array}{llll}
 & \cancel{12\,\text{flats}} & 8\,\text{longs} & 6\,\text{units} \\
1\,\text{block} & 2\,\text{flats} & & \\
\hline
1\,\text{block} & 8\,\text{flats} & 8\,\text{longs} & 6\,\text{units}
\end{array}
$$

As a result of the trading, we obtain 1 block 2 flats, 8 longs, and 6 units. This trade yields the fewest number of pieces, 1 + 2 + 8 + 6 = 17.

Other Number Base Systems

Quinary (base-five) system

One-Hand System	Base-Five Symbol	Base-Five Blocks
0 fingers	0_{five}	
1 finger	1_{five}	
2 fingers	2_{five}	
3 fingers	3_{five}	
4 fingers	4_{five}	
1 hand and 0 fingers	10_{five}	
1 hand and 1 finger	11_{five}	
1 hand and 2 fingers	12_{five}	
1 hand and 3 fingers	13_{five}	
1 hand and 4 fingers	14_{five}	
2 hands and 0 fingers	20_{five}	
2 hands and 1 finger	21_{five}	



Example 3

Convert 11244_{five} to base 10.

$5^4 = 625$	$5^3 = 125$	$5^2 = 25$	$5^1 = 5$	$5^0 = 1$
1	1	2	4	4

$$11244_5 = 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 4 \cdot 1$$

$$= 1 \cdot 625 + 1 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 4 \cdot 1$$

$$= 625 + 125 + 50 + 20 + 4$$

$$= 824$$

Base Two

Binary system – only two digits

Base two is especially important because of its use in computers.

One of the two digits is represented by the presence of an electrical signal and the other by the absence of an electrical signal.

Example 4a

Convert 10111_{two} to base ten.

$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
1	0	1	1	1

$$10111_2 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1$$

$$= 16 + 0 + 4 + 2 + 1$$

$$= 23$$

Example 4b

Convert 27 to base two.

[redacted]

or

2	27	
2	13	1
2	6	1
2	3	0
	1	1

$$27 = 11011_{\text{two}}$$

Base Twelve

Duodecimal system – twelve digits

Use *T* to represent a group of 10.

Use *E* to represent a group of 11.

The base-twelve digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *T*, and *E*.

Example 5a

Convert $E2T_{\text{twelve}}$ to base ten.

$$
\begin{aligned}
E2T_{12} &= 11\cdot 12^2 + 2\cdot 12^1 + 10\cdot 1 \\
&= 11\cdot 144 + 24 + 10 \\
&= 1584 + 24 + 10 \\
&= 1618
\end{aligned}
$$

Example 5b

Convert 1277 to base twelve.

$$\begin{array}{r|c|l} 144 & 1277 & 8 \\ \hline & -1152 & \\ 12 & 125 & T \\ \hline & -120 & \\ 1 & 5 & 5 \\ \hline & -5 & \\ \cline{2-2} & 0 & \end{array}$$

$$1277_{\text{ten}} = 8T5_{\text{twelve}}$$

Example 6

What is the value of *g* in $g36_{\text{twelve}} = 1050_{\text{ten}}$?

$$g \cdot 12^2 + 3 \cdot 12^1 + 6 \cdot 1 = 1050$$

$$144g + 36 + 6 = 1050$$

$$144g + 42 = 1050$$

$$144g = 1008$$

$$g = 7$$