Explicit Functions

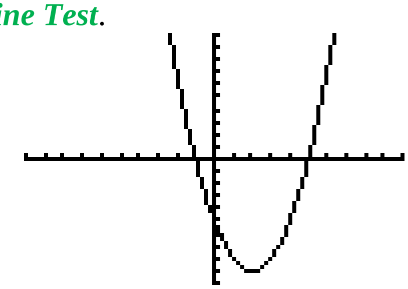
Most—if not all—of the equations relating one variable to another variable that you have examined in your mathematical career before today have been of the form $y = f(x)$. Each x-coordinate is paired with exactly one y-coordinate. For such relations, y is said to be an *explicit function of x*.

Example of an Explicit Function: $5 + y - x^2 = -4x$

(*) y can be isolated. Write the y-isolated form of this equation:

(**) Formulaically, y equals a single formula involving x.

(***) Graphically, the graph passes the ***Vertical Line Test***.



Implicit Functions

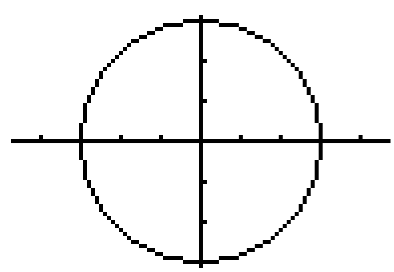
In the case of an implicit function, there is at least one x-coordinate (and possibly many more) that is paired with at least two different y-coordinates.

Example of an Implicit Function: $x^2 + y^2 = 9$

(*) You may or may not be able to isolate y. Try to isolate y:

(**) Formulaically, if you *can* isolate y, then y equals multiple formulas involving x.

(***) Graphically, the graph fails the ***Vertical Line Test***. For the given example, graph $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$ together.



The Derivative of an Explicit Function

(*) The limit definition of the derivative is how we have already defined the derivative function of an explicit function.

(**) The derivative function of an explicit function is itself an explicit function.

(***) Determine the derivative of $y = x^2 - 4x - 5$: $y' =$

(****) This derivative function give a formula for (an explicit function of) ___________

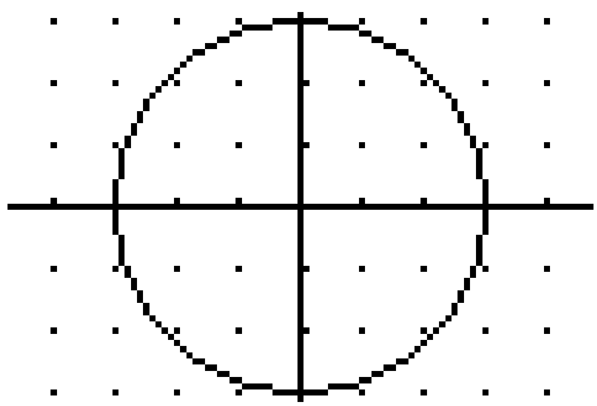
__________________ to the graph of __________________ at any *x*.

The Derivative of an Implicit Function

(*) We use a process called *Implicit Differentiation*, the main tool of which is the *Chain Rule*.

(**) The result is a formula (an implicit function) for the derivative $\frac{dy}{dx}$ (the slope of the tangent line). This formula may involve both *x* and *y*.

Exercise 1a: Consider the graph of $x^2 + y^2 = 9$. Estimate the slope of the tangent line at each of the points listed below.



(i) $\left(-1, \sqrt{8}\right)$

(ii) $\left(-1, -\sqrt{8}\right)$

(iii) (3 , 0)

(iv) (0 , 3)

Question: Why were the instructions for Exercise 1a **not** worded in the following way: "Estimate the slope of the tangent line at $x = -1$"?

Answer: *This is an ambiguous request, for there is more than one point on the graph whose x-coordinate is* -1 *. So, in order to specify a particular point (at which to draw a tangent line and estimate the slope there), we must also know the y-coordinate.*

Implicit Differentiation Step-by-Step

Exercise 1b: Determine the derivative of $x^2 + y^2 = 9$ implicitly.

Step 1: Rewrite the equation, replacing y with $y(x)$. →
This will help to remind you to use the *Chain Rule*.

Step 2: Take the derivative with respect to x. →
Use the *Chain Rule* (and other rules) as needed.

Step 3: Rewrite this equation with y in place of $y(x)$
and y' in place of $y'(x)$. →

Step 4: Isolate y', a notation equivalent to $\dfrac{dy}{dx}$. →

Exercise 1c: At what points is the derivative undefined? How does your answer correspond to the graph? (See page 2.)

Exercise 1d: Compute $\dfrac{d^2y}{dx^2}$ at $\left(-1, \sqrt{8}\right)$.

Exercise 2: For $2x^3 - y^4 = x^2y^2$, determine $\dfrac{dy}{dx}$.

Exercise 3: For $sin(xy) = x$, determine $\dfrac{dy}{dx}$.

Exercise 4: Determine the equation of the line tangent to the graph of $y^2 = \dfrac{x^2}{xy+20}$ at the point **(-4 , 1)**. *Make sure that you* ***check*** *that* ***(-4 , 1)*** *is indeed a point on the graph of this implicit function.*

Logarithmic Differentiation

Occasionally, it is convenient to use properties of logarithms before taking the derivative of some functions.

Exercise 5: Determine the derivative of $y = \frac{(x-3)^4}{\sqrt{x^2+1}}$. First, note that $y > 0$ and, hence, that $ln(y)$ is defined.

Exercise 6: Determine the derivative of $y = \sqrt{(x-1)(x-2)(x-3)}$.