The Arctangent Rule

Since (see page 176 in the text) the derivative of $f(x) = arctan(x)$ is $\dfrac{1}{x^2+1}$ then

$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$$

More generally, $$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

Exercise 1: Determine the following indefinite integral. $\int \dfrac{1}{x^2+16}\,dx$

Write this integral in the form $\int \dfrac{1}{u^2+a^2}\,du$ by letting $u = x$ and $a = 4$.

Note then that $du = dx$

Thus, $$\int \frac{1}{\underset{u^2}{x^2}+\underset{a^2}{16}}\,\overset{du}{dx} = \int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$$

Exercise 2: Determine the following indefinite integral. $\int \dfrac{1}{9x^2+16}\,dx$

Write this integral in the form $\int \dfrac{1}{u^2+a^2}\,du$ by letting $u = 3x$ and $a = 4$.

Note then that $du = 3\,dx$

Thus, $$\int \frac{1}{9x^2+16}\,dx = \frac{1}{3}\int \frac{1}{(\,3x\,)^2+(\,4\,)^2}\,3\,dx = \frac{1}{3}\int \frac{1}{u^2+a^2}\,du = \frac{1}{3}\cdot\frac{1}{a}\arctan\left(\frac{u}{a}\right) + c$$

$$= \frac{1}{3}\cdot\frac{1}{4}\arctan\left(\frac{3x}{4}\right) + C = \frac{1}{12}\arctan\left(\frac{3x}{4}\right) + C$$

Exercise 3: Determine the following indefinite integral. $\int \frac{x}{9x^4+50}\,dx$

$$\int \frac{x}{9x^4+50}\,dx = \int \frac{x}{(3x^2)^2+(\sqrt{50})^2}\,dx$$

Write this integral in the form $\int \frac{1}{u^2+a^2}\,du$ by letting $u = 3x^2$ and $a = \sqrt{50}$.

Note then that $du = 6x\,dx$

Thus, $\int \frac{x}{9x^4+50}\,dx = \frac{1}{6}\int \frac{6x}{(3x^2)^2+(\sqrt{50})^2}\,dx = \frac{1}{6}\int \frac{1}{u^2+a^2}\,du = \frac{1}{6}\,\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

$$= \frac{1}{6}\cdot\frac{1}{\sqrt{50}}\arctan\left(\frac{3x^2}{\sqrt{50}}\right)+C = \frac{1}{6\sqrt{50}}\arctan\left(\frac{3x^2}{\sqrt{50}}\right)+C$$

Exercise 4: Which of the following two integrals requires the arctangent rule, and which requires nothing more than basic u-substitution? Determine each indefinite integral.

$$\int \frac{x^3}{64x^8+4}\,dx$$

$u = 8x^4 \Rightarrow du = 32x^3\,dx$ Also, $a=2$

$$\int \frac{x^3}{64x^8+4}\,dx = \frac{1}{32}\int \frac{32x^3}{64x^8+4}\,dx$$

$$= \frac{1}{32}\int \frac{1}{u^2+a^2}\,du = \frac{1}{32}\cdot\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$= \frac{1}{32}\cdot\frac{1}{2}\arctan\left(\frac{8x^4}{2}\right)+C = \frac{1}{64}\arctan(4x^4)+C$$

$$\int \frac{x^7}{64x^8+4}\,dx$$

$u = 64x^8+4 \Rightarrow du = 512x^7\,dx$

$$\int \frac{x^7}{64x^8+4}\,dx = \frac{1}{512}\int \frac{512x^7}{64x^8+4}\,dx$$

$$= \frac{1}{512}\int \frac{du}{u} = \frac{1}{512}\ln|u|+C$$

$$= \frac{1}{512}\ln|64x^8+4|+C$$

The Arcsine Rule

Since (see page 176 in the text) the derivative of $f(x) = arcsin(x)$ is $\frac{1}{\sqrt{1-x^2}}$ then

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

More generally, $$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$$

Exercise 5: Determine the following indefinite integral. $\int \frac{x^5}{\sqrt{49-x^{12}}}\,dx$

Write this integral in the form $\int \frac{1}{\sqrt{a^2-u^2}}\,du$ by letting $u = x^6$ and $a = 7$.

Note then that $du = 6x^5\,dx$

Thus, $$\int \frac{x^5}{\sqrt{49-x^{12}}}\,dx = \frac{1}{6}\int \frac{6x^5}{\sqrt{49-x^{12}}}\,dx$$

$$= \frac{1}{6}\int \frac{1}{\sqrt{a^2-u^2}}\,du = \frac{1}{6}\arcsin\left(\frac{u}{a}\right) + c = \frac{1}{6}\arcsin\left(\frac{x^6}{7}\right) + C$$

Exercise 6: Determine the following definite integral. $\int_0^1 \frac{x+5}{\sqrt{9-x^2}}\,dx$

The numerator of the integrand looks like it will be **trouble**. However, since the numerator is a sum, we can rewrite the integrand as the sum of two fractions.

Thus, $\int_0^1 \frac{x+5}{\sqrt{9-x^2}}\,dx = \int_0^1 \frac{x}{\sqrt{9-x^2}}\,dx + \int_0^1 \frac{5}{\sqrt{9-x^2}}\,dx$

$u = 9-x^2 \Rightarrow du = -2x\,dx$ $\qquad$ $u = x \Rightarrow du = dx$ Also, $a = 3$

$x = 1 \Rightarrow u = 8$ $\qquad$ $x = 1 \Rightarrow u = 1$

$x = 0 \Rightarrow u = 9$ $\qquad$ $x = 0 \Rightarrow u = 0$

So, $\int_0^1 \frac{x+5}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{9-x^2}}\,dx + 5\int_0^1 \frac{1}{\sqrt{9-x^2}}\,dx$

$$= -\frac{1}{2}\int_9^8 \frac{du}{\sqrt{u}} + 5\int_0^1 \frac{1}{\sqrt{a^2-u^2}} = -\frac{1}{2}\int_9^8 u^{-1/2}\,du + 5\int_0^1 \frac{1}{\sqrt{a^2-u^2}}$$

$$= -\frac{1}{2}\cdot\frac{u^{1/2}}{1/2}\Bigg]_{u=9}^{u=8} + 5\arcsin\left(\frac{u}{a}\right)\Bigg]_{u=0}^{u=1} = -u^{1/2}\Bigg]_{u=9}^{u=8} + 5\arcsin\left(\frac{u}{3}\right)\Bigg]_{u=0}^{u=1}$$

$$= -\left(8^{1/2} - 9^{1/2}\right) + 5\left[\arcsin\left(\frac{1}{3}\right) - \arcsin\left(\frac{0}{3}\right)\right]$$

$$= -\sqrt{8} + 3 + 5\left[\arcsin\left(\frac{1}{3}\right) - 0\right] = -2\sqrt{2} + 3 + 5\arcsin\left(\frac{1}{3}\right)$$

The Arcsecant Rule

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + c$$

Exercise 7 (#12): Determine the following definite integral. $\int \frac{2}{x\sqrt{9x^2-25}}\,dx$

$$= \int \frac{2}{x\sqrt{(3x)^2-(5)^2}}\,dx$$

$u = 3x \Rightarrow du = 3dx$

Also, $a = 5$

$$= \frac{1}{3}\int \frac{2}{x\sqrt{(3x)^2-(5)^2}}\,3dx$$

We have made the appropriate adjustment for du, but the denominator is not in the form $u\sqrt{u^2-a^2}$.

We must multiply the denominator by 3, and, consequently, make another appropriate adjustment.

$$= \frac{3}{1}\cdot\frac{1}{3}\int \frac{2}{3x\sqrt{(3x)^2-(5)^2}}\,3dx$$

$$= 2\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{2}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + c = \frac{2}{5}\operatorname{arcsec}\left(\frac{|3x|}{5}\right) + c$$

Surgeon General's Warning: You may have to use your bag of algebraic and trigonometric tricks—or go to **WalMarth** and buy new ones—in order to be able to rewrite an integrand in a form that is recognizable for integration. We will look at a few of the tricks via some even exercises from the text.

SURGEON GENERAL'S WARNING: CALCULUS Causes Lung Cancer, Heart Disease, Emphysema, and May Complicate Pregnancy.

Exercise 8: $\int \frac{10}{4t^2-4t+5}\,dt$

Try *u*-substitution: $u = 4t^2-4t+5 \Rightarrow du = (8t-4)\,dt$

Trouble: The numerator is lacking, among other shortcomings, a power of t. And you cannot multiply in (and divide by) a variable.

Instead, start off by **completing the square** in the denominator:

$$4t^2-4t+5 = 4(t^2-t)+5 = 4\left(t^2-t+\left(-\tfrac{1}{2}\right)^2\right)-4\left(-\tfrac{1}{2}\right)^2+5$$

$$= 4\left(t-\tfrac{1}{2}\right)^2-4\cdot\tfrac{1}{4}+5 = \left[2\left(t-\tfrac{1}{2}\right)\right]^2+4 = (2t-1)^2+2^2$$

Thus, $\int \frac{10}{4t^2-4t+5}\,dt = \int \frac{10}{(2t-1)^2+2^2}\,dt$ Let $\boldsymbol{u} = 2t-1$ Then $\boldsymbol{du} = 2\,dt$

Also, let $\boldsymbol{a} = 2$

Then, $\int \frac{10}{(2t-1)^2+2^2}\,dt = \int \frac{5\cdot 2}{(2t-1)^2+2^2}\,dt = 5\int \frac{2}{(2t-1)^2+2^2}\,dt = 5\int \frac{1}{u^2+a^2}\,du$

du

$$= 5\cdot\frac{1}{a}\cdot\arctan\left(\frac{u}{a}\right)+C = 5\cdot\frac{1}{2}\cdot\arctan\left(\frac{2t-1}{2}\right)+C$$

$$= \frac{5}{2}\arctan\left(\frac{2t-1}{2}\right)+C$$