Assoc. Professors Bob and Lisa Brown

Use the completed handout to complete the notes.

Revisiting Indeterminate Forms of Limits

Suppose that, upon evaluating the limit $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ by direct substitution, we get $\frac{0}{0}$. In Calculus 1 courses that don't cover *L'Hôpital's Rule* (in mine we do), students are occasionally able to genetically alter the molecular structure of the functions *f* and *g*—that is, *factor* or apply some other algebraically invasive procedure. And when no algebraic maneuver works, you need a table to evaluate or approximate the limit.

We will now study a powerful tool called *L'Hôpital's Rule*. Before we do that, we will first need to revisit some old ideas and explore some new ones, which will be used below to prove *L'Hôpital's Rule*.

Equation of a Tangent Line

The *Point-Slope Form* of the equation of a "**random**" line with slope *m* that passes through $(x_0\ ,\ f(x_0))$, where *f* is some function of *x*, is:

The *Point-Slope Form* of the equation of the line **tangent** to $y = f(x)$ at $x = x_0$ is:

We'll also assume that *f* and *g* are both differentiable at x_0, which means that (see Chapter 4, Section 8, starting on page 271 in the textbook) **for all *x* near x_0,**

$f(x) \approx$ and $g(x) \approx$

Thus, $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{\lim_{x\to x_0} [f(x_0) + f'(x_0)(x-x_0)]}{\lim_{x\to x_0} [g(x_0) + g'(x_0)(x-x_0)]} =$

$= \frac{f'(x_0)}{g'(x_0)}$

We'll also assume that f' and g' are continuous at x_0, which means that

$\lim_{x\to x_0} f'(x) =$ and $\lim_{x\to x_0} g'(x) =$

Thus, $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} = \frac{\lim_{x\to x_0} f'(x)}{\lim_{x\to x_0} g'(x)} =$

L'Hôpital's Rule

Watch this video for an explanation of the proof of *L'Hôpital's Rule*:

Theorem: With the following three assumptions/hypotheses:

1. $\lim_{x \to x_0} f(x) =$ ☐ and $\lim_{x \to x_0} g(x) =$ ☐,
2. f and g are ☐ at x_0, and
3. f' and g' are ☐ at x_0,

then $\lim_{x \to x_0} \frac{f(x)}{g(x)} =$ ☐

Exercise 1: Example with **indeterminate form** $\frac{0}{0}$. Evaluate $\lim_{x \to 2} \frac{3x^2 - 12}{x - 2}$.

You may recall this example from Calculus 1, Handout 2.2, Exercise 2, pages 1 – 3.
I mean, what else would you be doing since last semester other than re-reading and memorizing the Calc 1 handouts I gave you?

In each problem, you should check that **all three** hypotheses of *L'Hôpital's Rule* are satisfied. However, in just about every problem that we'll encounter, the second and third hypotheses will be met; we'll just need to check the first—that direct substitution renders an indeterminate form.

Direct Substitution:

L'Hôpital's Rule:

Watch this video for an example of a similar problem to Exercise 2:

Exercise 2: Another example with **indeterminate form** $\frac{0}{0}$. Evaluate $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$.

Direct Substitution:

L'Hôpital's Rule:

Watch this video for an example of a similar problem to Exercise 3:

Exercise 3: Example with indeterminate form $\frac{\infty}{\infty}$. Evaluate $\lim\limits_{x\to\infty}\frac{\ln(x)}{x}$.

$\frac{\lim\limits_{x\to\infty}\ln(x)}{\lim\limits_{x\to\infty}x}$ =

L'Hôpital's Rule:

Watch this video for an example of a similar problem to Exercise 4:

Exercise 4: Example applying L'Hôpital's Rule more than once. Evaluate $\lim\limits_{x\to-\infty}\frac{x^2}{e^{-x}}$.

$\frac{\lim\limits_{x\to-\infty}x^2}{\lim\limits_{x\to-\infty}e^{-x}}$ =

L'Hôpital's Rule:

Watch this video for an example of a similar problem to Exercise 5:

Exercise 5: Example with indeterminate form $0\cdot\infty$. Evaluate $\lim\limits_{x\to\infty}e^{-x}\sqrt{x}$.

Exercise 6: Example with indeterminate form 1^{∞}. Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$.

Watch this video for an example of a similar problem to Exercise 7:

Exercise 7: Example with **indeterminate form** 0^0. Evaluate $\lim_{x\to 0^+} (\sin(x))^x$.

Assume that the limit exists; call the limit value y. Thus, $y = \lim_{x\to 0^+} (\sin(x))^x$.

Since the limit value is positive—because $sin(x)$ is positive for small, positive values of x—we can apply the natural log to both sides.

$$y = \lim_{x\to 0^+} (\sin(x))^x$$

So, $ln(y) = \mathbf{0}$, implying that $y =$ ☐. Thus, $\lim_{x\to 0^+} (\sin(x))^x =$ ☐.

Watch this video for an example of a similar problem to Exercise 8:

Exercise 8: Example with **indeterminate form** $\infty - \infty$. Evaluate $\lim_{x\to 1^+}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right)$.

We first must write $\frac{1}{\ln(x)} - \frac{1}{x-1}$ in the form $\frac{f(x)}{g(x)}$.

$$\frac{1}{\ln(x)} - \frac{1}{x-1} = \frac{1}{\ln(x)} \qquad - \frac{1}{(x-1)} \qquad =$$

Now, $\lim_{x\to 1^+}\frac{x-1-\ln(x)}{(x-1)\ln(x)} =$

Apply *L'Hôpital's Rule*: $\lim_{x\to 1^+}\frac{[x-1-\ln(x)]'}{[(x-1)\ln(x)]'} =$

Multiply by : $\lim_{x\to 1^+}\frac{\left(1-\frac{1}{x}\right)}{\left[\ln(x)+(x-1)\frac{1}{x}\right]} =$

Apply *L'Hôpital's Rule* again:

Exercise 9: Find the ***error*** in the following sequence of steps.

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x} = \lim_{x\to 0}\frac{\left(e^{2x}-1\right)'}{\left(e^x\right)'} = \lim_{x\to 0}\frac{2e^{2x}}{e^x} = \lim_{x\to 0}\frac{2\left(e^x\right)^2}{e^x} = \lim_{x\to 0} 2e^x = 2$$ (**incorrect**)

Note: See the textbook for examples involving other indeterminate forms: 0^0 and $\infty - \infty$.

More Videos and Resources:

Video 1

Video 2

Textbook Exercises: Section 5.6

Problems: 3, 7, 11, 19, 23, 29, 33, 37, 45, 51, 55, 71, 73

Textbook Exercises Videos: Section 5.6

Problem 7

Problem 29

Problem 51

Textbook PowerPoints Slides: Section 5.6

View a summary of the textbook reading in PowerPoint form.

Hyperlinks:

- Completed Handout: https://cwoer.ccbcmd.edu/math/math252/m252c5s6sol.pdf
- Video of the proof of L'Hôpital's Rule: https://cwoer.ccbcmd.edu/math/math252/m252c5s6sol.pdf
- Video similar to Exercise 2: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v01038a.html
- Video similar to Exercise 3: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v01039a.html
- Video similar to Exercise 4: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v02488a.html
- Video similar to Exercise 5: https://youtu.be/8kWXBoP439w
- Video similar to Exercise 7: https://youtu.be/kEnwac_9lyg
- Video similar to Exercise 8: https://youtu.be/g_sX1yd3jmI
- Video Resource 1: https://youtu.be/Sp0G-VggAoU
- Video Resource 2: https://www.khanacademy.org/math/calculus-home/derivative-applications-calc/lhopitals-rule-calc/v/introduction-to-l-hopital-s-rule
- Textbook Exercises Video 5.6 Problem 7: https://youtu.be/ADDqgwo6a0c
- Textbook Exercises Video 5.6 Problem 29: youtu.be/SLIgPlJ2tDw
- Textbook Exercises Video 5.6 Problem 51: youtu.be/5qwaGcTR5Pw
- Textbook Summary PowerPoint Section 5.6: https://cwoer.ccbcmd.edu/math/math252/Math252Section0506.pptx