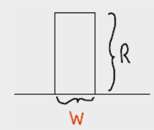
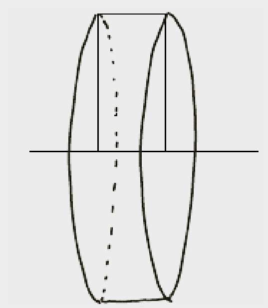
Use this link for the Blank Handout to take notes.

Volume of a Solid of Revolution: Using the Disk Method

Another application of the definite integral is the computation of the volume of a particular type of three-dimensional solid, called a *solid of revolution*. A solid of revolution is obtained by revolving a region in the plane about a line. This line is called the *axis of revolution*.

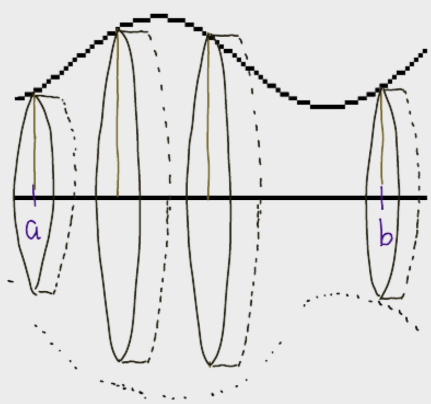


Revolving this rectangle, we get a disk—a hockey puck. Compute the volume of the disk:



- cross-sectional area is the area of a circle $= \pi R^2$
- width $= w$ Therefore, volume $= \pi R^2 w$

Revolving a More General Region



The volume of a solid of revolution is approximated by the sum of the volumes of the n disks.

The radius of each disc is $R_i = f(x_i)$ for some x_i in each subinterval of the partition of $a \le x \le b$, since the axis of revolution is the x-axis in this drawing.

The width of each disk is Δx.

The *approximate* volume of the solid of revolution is

$$\pi R_1^2 \Delta x + \pi R_2^2 \Delta x + \dots + \pi R_n^2 \Delta x = \sum_{i=1}^{n} \pi R_i^2 \Delta x$$

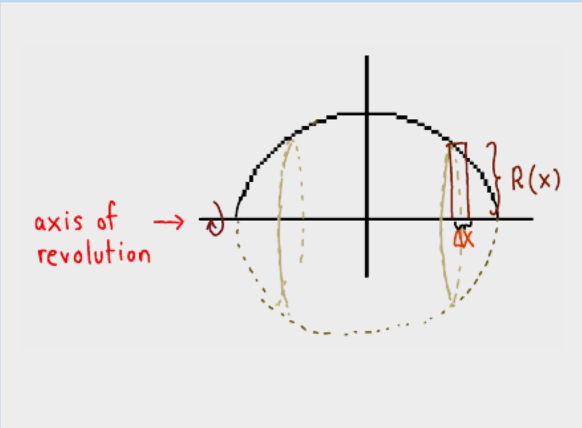
The *exact* volume of the solid of revolution is

$$\lim_{n \to \infty} \sum_{i=1}^{n} \pi R_i^2 \Delta x = \pi \int_a^b [R(x)]^2 dx$$

Note: In the case that the **x-axis is the axis of revolution**, then $R(x) =$ $f(x)$.

Watch this video for an example of determining the volume of a solid of revolution.

Exercise 1: Determine the volume of the solid of revolution formed by revolving the region bounded by $f(x) = \sqrt{\cos(x)}$, the x-axis, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, about the x-axis.



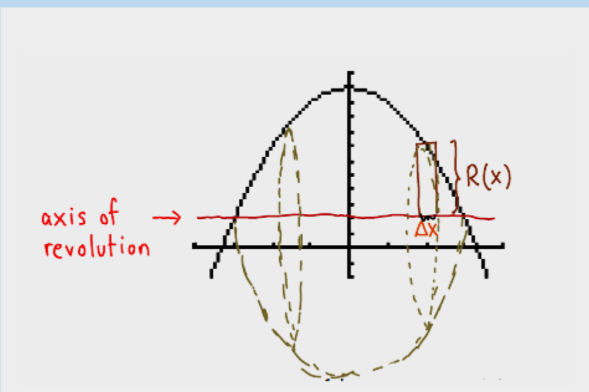
Note: In this exercise, $R(x) =$ $f(x) = \sqrt{\cos(x)}$ since the x-axis is the axis of revolution.

$$\text{Volume} = \pi \int_{-\pi/2}^{\pi/2} \left[\sqrt{\cos(x)}\right]^2 dx = \pi \int_{-\pi/2}^{\pi/2} \cos(x)\, dx$$

$$= \pi \sin(x) \Big|_{x=-\pi/2}^{x=\pi/2} = \pi\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)\right] = \pi(1 - -1)$$

$$= 2\pi$$

Exercise 2: Determine the volume of the solid of revolution formed by revolving the region bounded by $f(x) = 11 - x^2$ and $y = 2$ about the line $y = 2$.



What is the axis of revolution? $y = 2$

What are a and b? That is, for what x-values does f intersect $y = 2$? To answer this, we solve

$$11 - x^2 = 2$$
$$9 = x^2$$
$$\pm 3 = x \qquad a = -3,\; b = 3$$

Note that, in this exercise $R(x) \ne f(x)$ since the x-axis is *not* the axis of revolution.

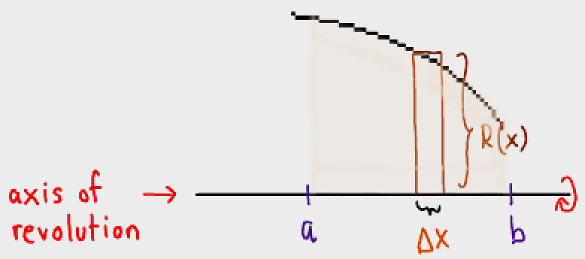
In fact, the radius is two less than the height of the graph of $f(x)$. Thus $R(x) = f(x) - 2$.

$$\text{Vol.} = \pi \int_a^b [R(x)]^2\, dx = \pi \int_{-3}^{3} [f(x) - 2]^2\, dx = \pi \int_{-3}^{3} [11 - x^2 - 2]^2\, dx = \pi \int_{-3}^{3} [9 - x^2]^2\, dx$$

$$= \pi \int_{-3}^{3} (x^4 - 18x^2 + 81)\, dx = \pi\left(\frac{x^5}{5} - 6x^3 + 81x\right)\Bigg|_{x=-3}^{x=3} = \pi\left(\frac{1296}{10} - \frac{-1296}{10}\right) = \frac{1296\pi}{5}$$

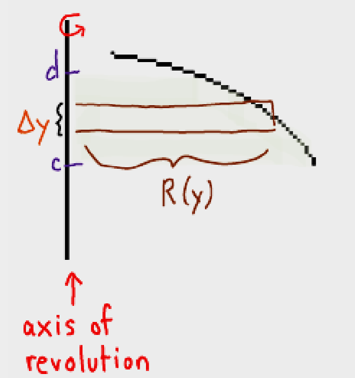
When you determine the volume of a solid of revolution using the *disk method*, your diagram will resemble one of the two following diagrams. **Note that with the *disk method*, the representative rectangle that you draw will be *perpendicular* to the axis of revolution**.

Horizontal Axis of Revolution



$$\text{volume} = \pi \int_a^b [R(x)]^2\,dx$$

Vertical Axis of Revolution



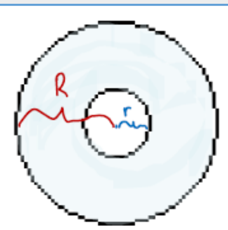
$$\text{volume} = \pi \int_c^d [R(y)]^2\,dy$$

Annulus

The disk method can be generalized to a method called the *washer method*. Before we discuss the washer method, we must review the definition of an annulus.

Definition: An annulus is a circular region in the plane that has had a smaller, inner, centered, circular region removed.

Watch this video explanation of an annulus.



The annulus is the shaded region.

What is the area of an annulus?

Area = area of outer circle minus area of inner circle

$$= \pi R^2 - \pi r^2$$

$$= \pi\left(R^2 - r^2\right)$$

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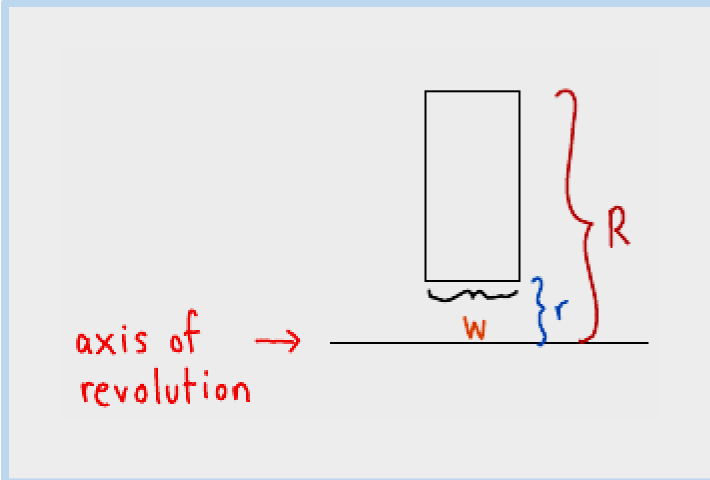
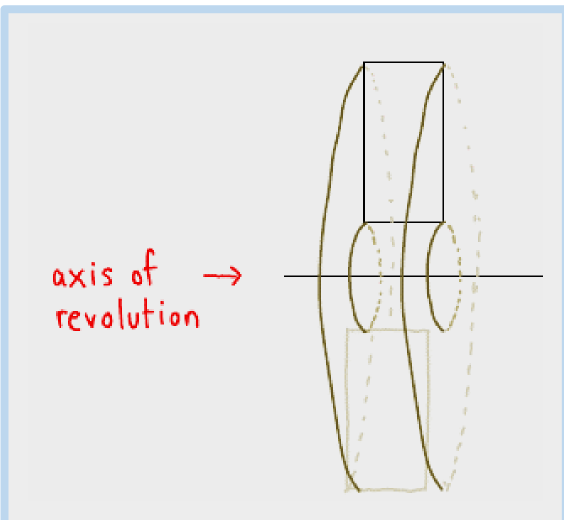
Volume of a Solid of Revolution: Using the Washer Method

The disc method can be generalized, or extended, to cover solids of revolution with holes, by replacing the representative disc with a representative washer.



Revolving this rectangle, we get a washer. Compute the volume of the washer:



- cross-sectional area is the area of an annulus $= \pi(R^2 - r^2)$
- width $= w$ Therefore, volume $= \pi(R^2 - r^2)\,w$

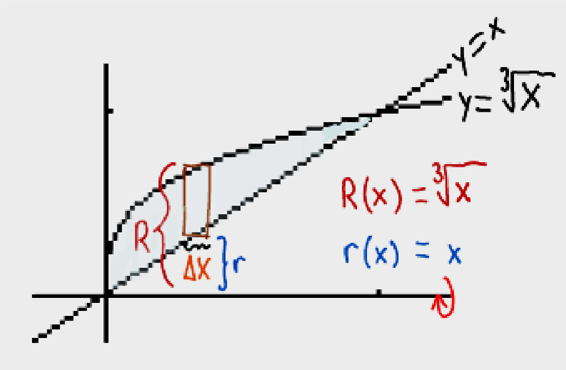
Note that with the *washer method*, as with the disc method, the representative rectangle that you draw is *perpendicular* to the axis of revolution.

In general, by the washer method, the volume of the solid of revolution is

$$\pi \int_a^b \left[(R(x))^2 - (r(x))^2 \right] dx \quad \text{or} \quad \pi \int_c^d \left[(R(y))^2 - (r(y))^2 \right] dy$$

Watch this video example of the washer method.

Exercise 3: Determine the volume of the solid of revolution formed by revolving the region in the first quadrant bounded by the graphs of $y = x$ and $y = \sqrt[3]{x}$ about the x-axis.



First, determine a and b by solving $x = \sqrt[3]{x}$

$$(x)^3 = \left(\sqrt[3]{x}\right)^3$$

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x(x+1)(x-1) = 0 \qquad x = -1, 0, 1$$

Hence, $a = 0$ and $b = 1$

Note that $R(x) = \sqrt[3]{x}$ and $r(x) = x$

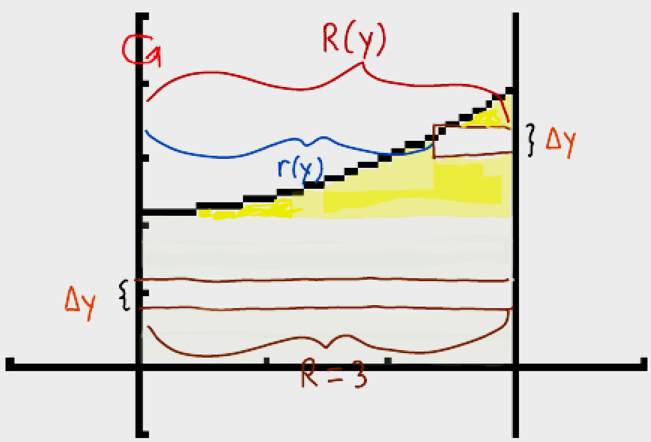
Thus, the volume $= \pi \displaystyle\int_0^1 \left[(R(x))^2 - (r(x))^2 \right] dx = \pi \int_0^1 \left[\left(\sqrt[3]{x}\right)^2 - (x)^2 \right] dx$

$$= \pi \int_0^1 \left(x^{2/3} - x^2\right) dx = \pi \left(\frac{3}{5} x^{5/3} - \frac{x^3}{3} \right) \Bigg|_{x=0}^{x=1} = \pi \left(\frac{3}{5} - \frac{1}{3} - (0 - 0) \right) = \frac{4\pi}{15}$$

Watch a video example with integration with respect to y.

Watch a video of Exercise 4.

Exercise 4: Determine the volume of the solid of revolution formed by revolving the region bounded by the graphs of $f(x) = 11 + x^2$, the x-axis, the y-axis, and $x = 3$ about the y-axis.



$R(y) = 3$

$r(y)$ = distance from y-axis to the graph
= x-coordinate (of the function)
= x

Thus, solve $y = 11 + x^2$ for x.

$$y - 11 = x^2$$

$$\pm\sqrt{y-11} = x$$

$x = +\sqrt{y-11}$ since all x-coordinates on the given graph are to the right of the y-axis.

$$\text{Volume} = \pi\int_0^{11} R^2\,dy + \pi\int_{11}^{20}\left[(R(y))^2 - (r(y))^2\right]dy$$

$$= \pi\int_0^{11} 3^2\,dy + \pi\int_{11}^{20}\left[(3)^2 - (\sqrt{y-11})^2\right]dy$$

$$= \pi\int_0^{11} 9\,dy + \pi\int_{11}^{20}\left[9 - (y-11)\right]dy$$

$$= \pi\int_0^{11} 9\,dy + \pi\int_{11}^{20}(20-y)\,dy = \pi\cdot 9y\Big|_{y=0}^{y=11} + \pi\left(20y - \frac{y^2}{2}\right)\Big|_{y=11}^{y=20}$$

$$= \pi(99-0) + \pi(200-159.5) = 99\pi + 40.5\pi = 139.5\pi$$

More Example Videos:

Volume of Revolution about a Vertical Line Washer Method video

Volume of Revolution Washer Method Example 1 video

Volume of Revolution Washer Method Example 2 video

Volume of Revolution Washer Method Example 3 video

Textbook Exercises: Section 7. 2

Problems: 5-37 odd, 47, 57, 61, 62, 65, 69a, 73

Textbook Exercises Videos: Section 7. 2

Problem 9

Problem 17

Problem 23

Problem 37

Textbook PowerPoints Slides: Section 7. 2

View a summary of the textbook reading in PowerPoint form.

Hyperlinks:

- Blank Handout: https://cwoer.ccbcmd.edu/math/math252/m252c7s2.pdf
- Solid of Revolution video: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v01023a.html
- Solid of Revolution Example video: https://www.youtube.com/embed/DRFyNHdVgUA?r=0
- Annulus video: https://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/video_wrapper.html?filename=v01025a
- Washer Method video: https://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/video_wrapper.html?filename=v01026a
- Integration with Respect to y video: https://www.youtube.com/embed/43AS7bPUORc?r=0
- Exercise 4 video: https://www.youtube.com/embed/iEv-LAVY3EM?r=0
- Volume of Revolution about a Vertical Line Washer Method : http://patrickjmt.com/volumes-of-revolution-volumes-about-vertical-lines-using-washers/
- Volume of Revolution Washer Method Example 1:http://patrickjmt.com/volumes-of-revolution-diskwashers-example-1/
- Volume of Revolution Washer Method Example 2: http://patrickjmt.com/volumes-of-revolution-diskwashers-example-2/

- Volume of Revolution Washer Method Example 3: http://patrickjmt.com/volumes-of-revolution-diskwashers-example-3/
- Textbook Exercises Video 7.2 Problem 9: youtu.be/6Udn9Cj4AdQ
- Textbook Exercises Video 7.2 Problem 17: youtu.be/burfVO5OnJ4
- Textbook Exercises Video 7.2 Problem 23: youtu.be/Rob9-qVo1ZM
- Textbook Exercises Video 7.2 Problem 37: youtu.be/NJuNkOqSyCo
- Textbook PowerPoint 7.2: https://cwoer.ccbcmd.edu/math/math252/Math252Section0702.pptx