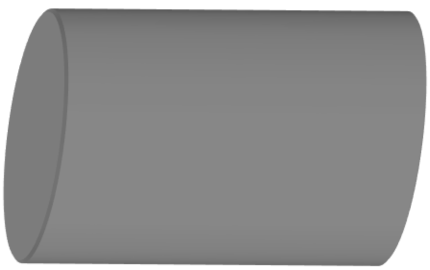
Assoc. Professors Bob and Lisa Brown

Use this link for the blank handout to take notes.

Volume of a Solid of Revolution Using the Shell Method

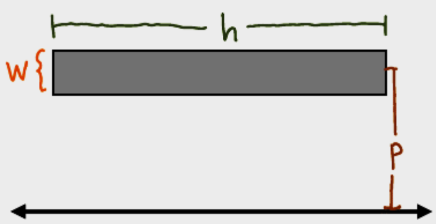
In this section, we will look at another method for determining the volume of a solid of revolution, called the *shell method*. It makes use of cylindrical shells, and it is sometimes easier to use than the disc and washer methods. Watch this video about the shell method.

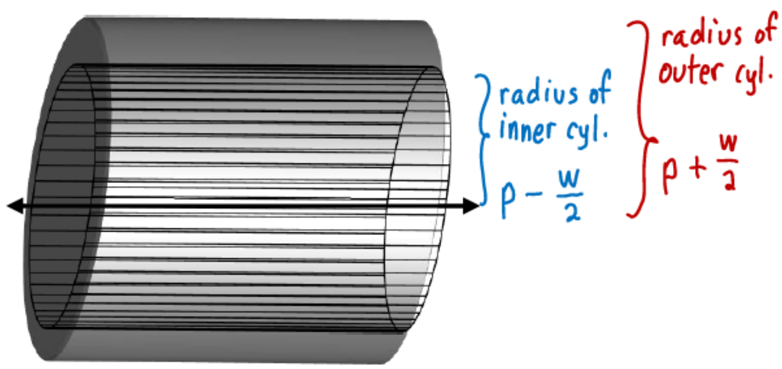
First of all, recall the formula for the volume of a cylinder.



$$\text{Volume} = \text{cross sectional area} \times \text{height}$$
$$= \pi r^2 \times h$$
$$= \pi r^2 h$$

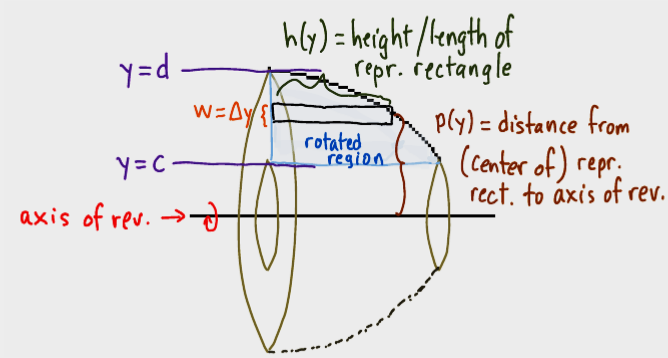
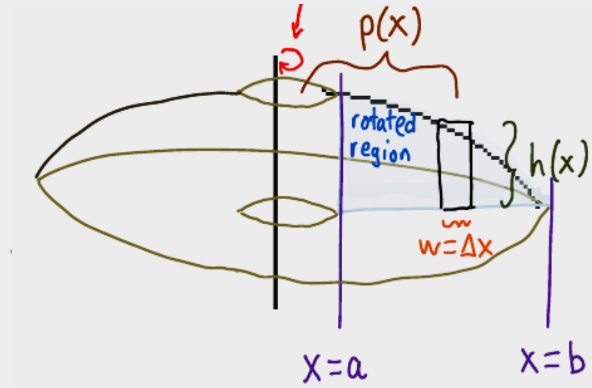
Now, consider the following rectangle, drawn parallel to the axis of revolution, with width *w*, height (length) *h*, and distance *p* from the axis of revolution to the center of the rectangle. We revolve the rectangle about the axis of revolution and form a *cylindrical shell*.





$$\text{Volume of cylindrical shell} = \text{volume of outer cylinder} \quad \text{minus} \quad \text{volume of inner cylinder}$$
$$= \pi\left(p+\frac{w}{2}\right)^2 h - \pi\left(p-\frac{w}{2}\right)^2 h$$
$$= \pi h\left[\left(p+\frac{w}{2}\right)^2 - \left(p-\frac{w}{2}\right)^2\right]$$
$$= \pi h\left[\left(p^2+2p\frac{w}{2}+\frac{w^2}{4}\right)-\left(p^2-2p\frac{w}{2}+\frac{w^2}{4}\right)\right]$$
$$= \pi h\left[\left(p^2+pw+\frac{w^2}{4}\right)-\left(p^2-pw+\frac{w^2}{4}\right)\right]$$
$$= \pi h\left[p^2+pw+\frac{w^2}{4}-p^2+pw-\frac{w^2}{4}\right]$$
$$= \pi h\left[2pw\right]$$
$$= 2\pi phw$$

When you determine the volume of a solid of revolution using the *shell method*, your diagram will resemble one of the two following diagrams. **Note that with the *shell method*, the representative rectangle that you draw will be *parallel* to the axis of revolution.**

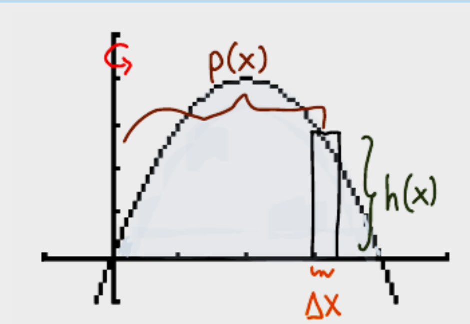
Horizontal Axis of Revolution	Vertical Axis of Revolution
	
Volume = $2\pi \int_c^d p(y)\,h(y)\,dy$	Volume = $2\pi \int_a^b p(x)\,h(x)\,dx$

Watch this video of an example of the shell method.

Exercise 1: Use the shell method to determine the volume of the solid of revolution formed by revolving the region bounded by $f(x) = -(x-2)^2 + 4$ and the x-axis about the y-axis.

Steps

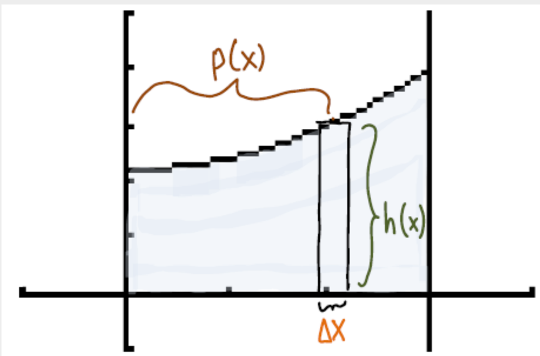
1. Draw the representative rectangle parallel to the axis of revolution
2. Label the width appropriately. In *this* exercise, the width of the representative rectangle is Δx
   This tells you to write p and h as functions of x.



3. Label p and h.
   In *this* exercise, $p(x)$ = x , since, in this exercise, p is the distance from the y-axis to the curve.
   In *this* exercise, $h(x)$ = y , since, in this exercise, h is the distance from the x-axis to the curve... And $y = f(x)$; so, $h(x) = f(x) = -(x-2)^2 + 4$
   Therefore, volume = $2\pi \int_a^b p(x)\,h(x)\,dx = 2\pi \int x\left[-(x-2)^2+4\right]dx = \frac{128\pi}{3}$ by the T.I.-89

Recalling Exercise 4 from page 6 of Handout 7.2, consider Exercise 2 below. In Exercise 2, we will re-compute the volume of the solid of revolution, but this time we will use the shell method instead of using both the disk and washer methods as we did in Handout 7.2. In 7.2, we had to use both the disk and the washer methods, and we had to write two integrals. Using the shell method in this case will require only one integral, which will require less effort.

Exercise 2: Determine the volume of the solid of revolution formed by revolving the region bounded by the graphs of $f(x) = 11 + x^2$, the x-axis, the y-axis, and $x = 3$ about the y-axis.

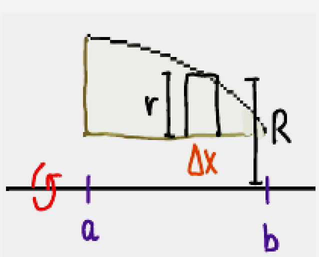
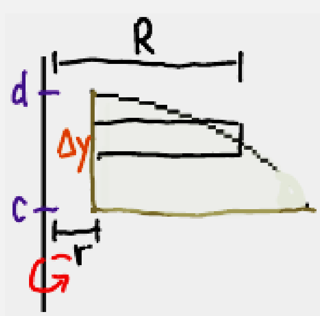
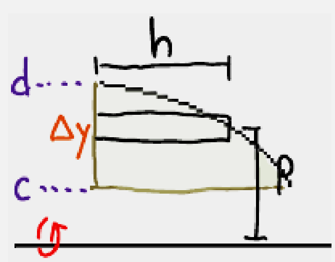
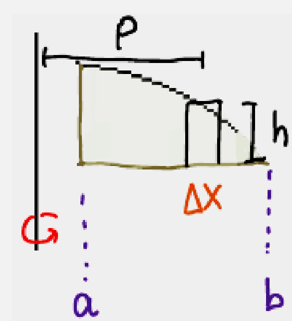


$h(x)$ = distance from x-axis to curve $= y$
$= f(x) = 11 + x^2$

$p(x)$ = distance from y-axis to curve $= x$

Volume = $2\pi \int_0^3 x(11+x^2)\,dx$

A Comparison of the Washer/Disk Method with the Shell Method

Washer/Disk Method		Shell Method	
			
The representative rectangle is **perpendicular** to the axis of revolution.	The representative rectangle is **perpendicular** to the axis of revolution.	The representative rectangle is **parallel** to the axis of revolution.	The representative rectangle is **parallel** to the axis of revolution.
Here, the axis of revolution is **horizontal.**	Here, the axis of revolution is **vertical.**	Here, the axis of revolution is **horizontal.**	Here, the axis of revolution is **vertical.**
Volume = $\pi\int_a^b [(R(x))^2-(r(x))^2]\,dx$	Volume = $\pi\int_c^d [(R(y))^2-(r(y))^2]\,dy$	Volume = $2\pi\int_c^d p(y)h(y)\,dy$	Volume = $2\pi\int_a^b p(x)h(x)\,dx$

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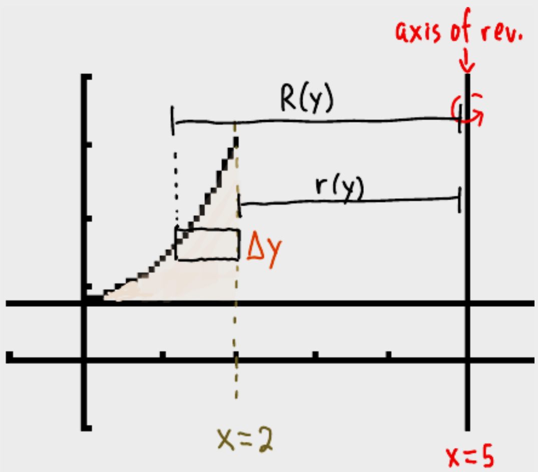
Watch this video comparing the washer and shell methods.

Exercise 3: Determine the volume of the solid of revolution formed by revolving the region bounded by the graphs of $y = x^3 + 2x + 4$, $y = 4$, and $x = 2$ about the line $x = 5$.

First, try to determine the volume using the *washer method*.



$R(y) = 5 - x$

$r(y) = 3$

The difficulty is that we would have to write

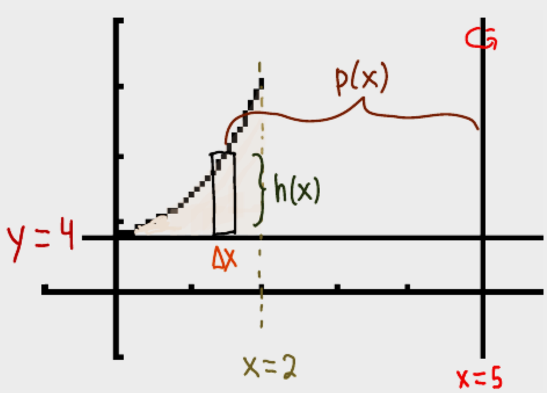
$y = x^3 + 2x + 4$ as x in terms of y

in order to have a formula for $R(y) = 5 - x$ in terms of y.

In other words, we would have to solve

$y = x^3 + 2x + 4$ for x. Try to solve for x!

Now, instead, determine the volume using the *shell method*.



$h(x) = f(x) - 4 = x^3 + 2x + 4 - 4 = x^3 + 2x$

$p(x) = 5 - x$

$$\text{Volume} = 2\pi \int_0^2 (5 - x)(x^3 + 2x)\,dx$$

$$= \frac{848\pi}{15} \text{ by the T.I.-89}$$

More Example Videos:

Shell method for rotating around vertical line video

Volume of Revolution The Shell Method about the y-axis video

Volume of Revolution The Shell Method about the x-axis video

The Shell Method NOT about x or y axis video

Textbook Exercises: Section 7. 3

Problems: 3-31 odd, 35, 37, 38, 39, 49, 55, 59

Textbook Exercises Videos: Section 7. 3

Problem 7

Problem 17

Problem 23

Problem 27

Textbook PowerPoints Slides: Section 7. 3

View a summary of the textbook reading in PowerPoint form.

Hyperlinks:

- Blank Handout: https://cwoer.ccbcmd.edu/math/math252/m252c7s3.pdf
- Shell Method Introduction Video: https://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/video_wrapper.html?filename=v01404a
- Video of an example of the shell method: http://patrickjmt.com/volumes-of-revolution-cylindrical-shells/
- Video comparing the washer and shell method: https://www.youtube.com/embed/ZyFaaKhNPXo?r=0
- Shell method for rotating around vertical line: https://www.youtube.com/embed/6Ozz3J-LRrY?r=0
- Volume of Revolution The Shell Method about the y-axis: https://www.youtube.com/embed/3B2YQbEzshg?r=0
- Volume of Revolution The Shell Method about the x-axis: https://www.youtube.com/embed/pCMkHkprN0I?r=0
- The Shell Method NOT about x or y axis: https://www.youtube.com/embed/lp3_rmjbxZ8?r=0
- Textbook Exercises Video 7.3 Problem 7: youtu.be/KAAkpcugZC0
- Textbook Exercises Video 7.3 Problem 17: youtu.be/dzc9V4PKi6U
- Textbook Exercises Video 7.3 Problem 23: youtu.be/qwFgJPPRJRg
- Textbook Exercises Video 7.3 Problem 27: youtu.be/_5PrYwoSfN4
- Textbook Summary PowerPoint Section 7.3: https://cwoer.ccbcmd.edu/math/math252/Math252Section0703.pptx